Computer-based Control Systems

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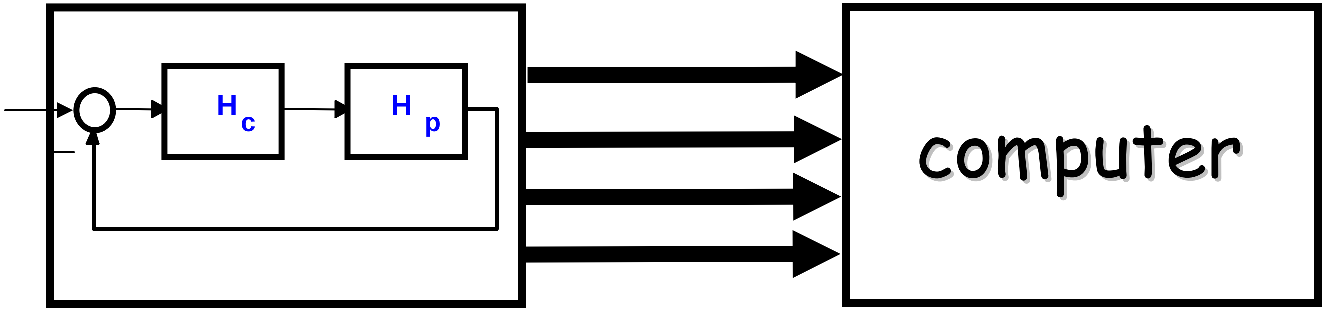
Contents

- Computers in control systems
  - Configurations
- Sampling, discretisation, real time
  - AD/DA conversion, ZOH
- Mathematics
  - Impuls sampling
  - Nyquist frequency (Shannon)
  - z-transform
- Euler, Tustin, real z-transform

Why computers

- Advantages
  - General purpose hardware
  - Flexibility
  - More functionality
- Disadvantages
  - Sampled data:
    - Stability
    - High frequencies in control signal

Configurations

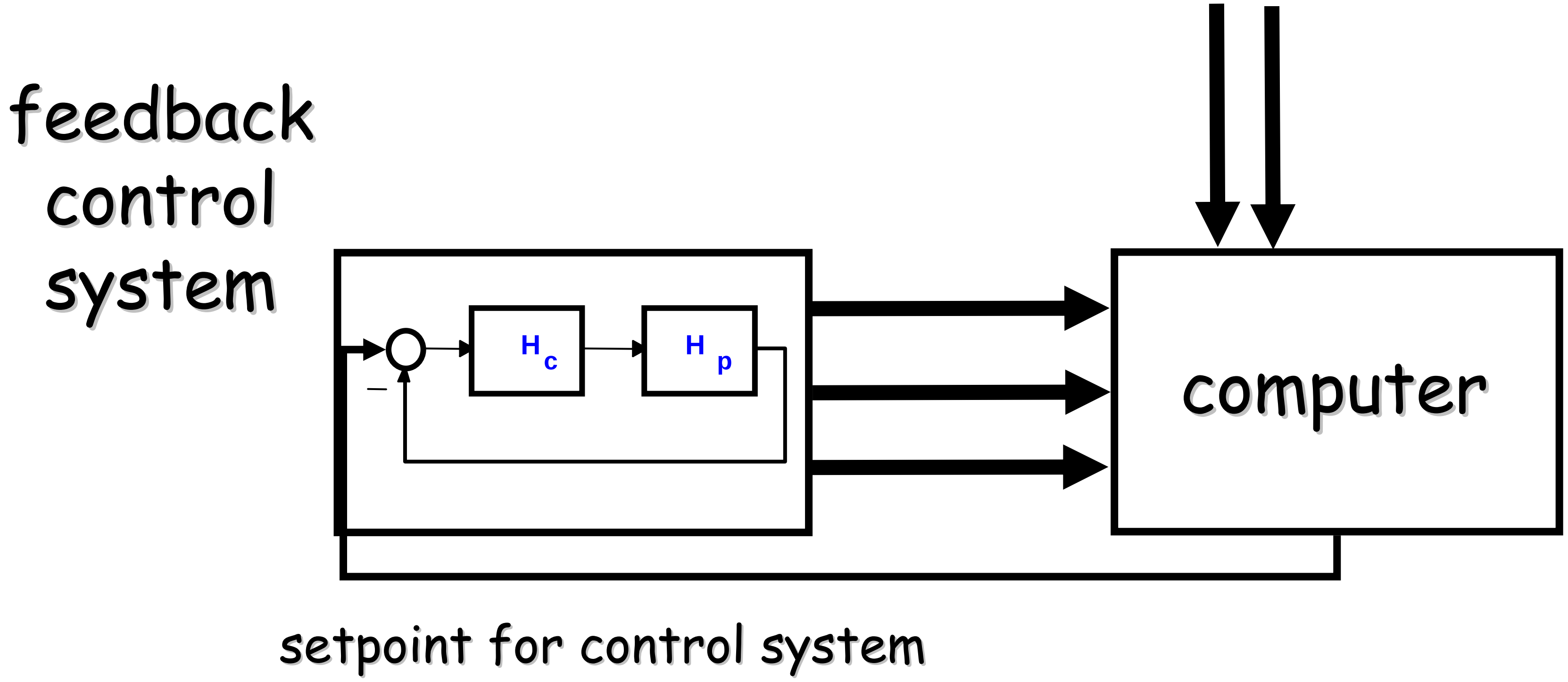


feedback
control
system

data logging

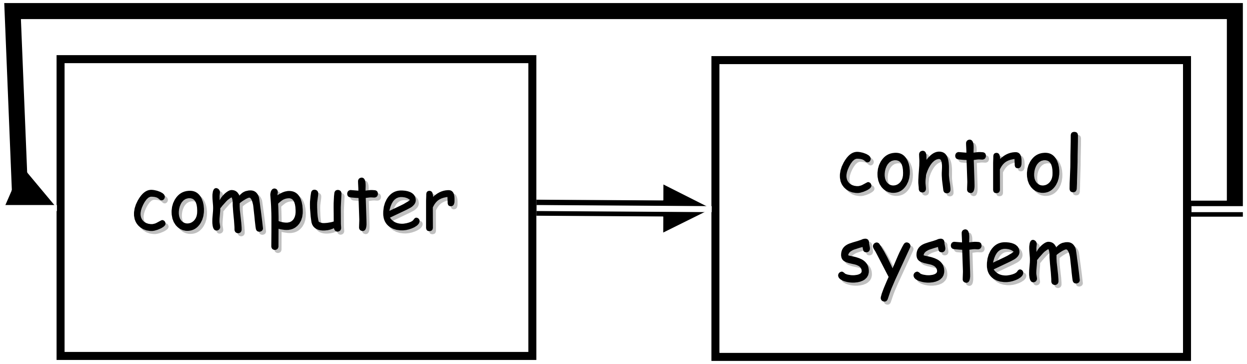
computer = add on

Configurations



optimization

Configurations



Direct Digital Control

computer part of the control loop

PDP-9 (1970), PDP8, 11

- 64 kB memory
- DEC-tapes
- Boot from paper tape
- AD/DA
- € 500.000

Analogue versus digital

- Process: analogue:
  - mostly a continuous-time process
- Computer: data:
  - Data are measured
  - Computations take some time
  - Data are send to the outside world
- Computer is digital
  - Means also 'limited accuracy'

Computer view

Seen from the computer:

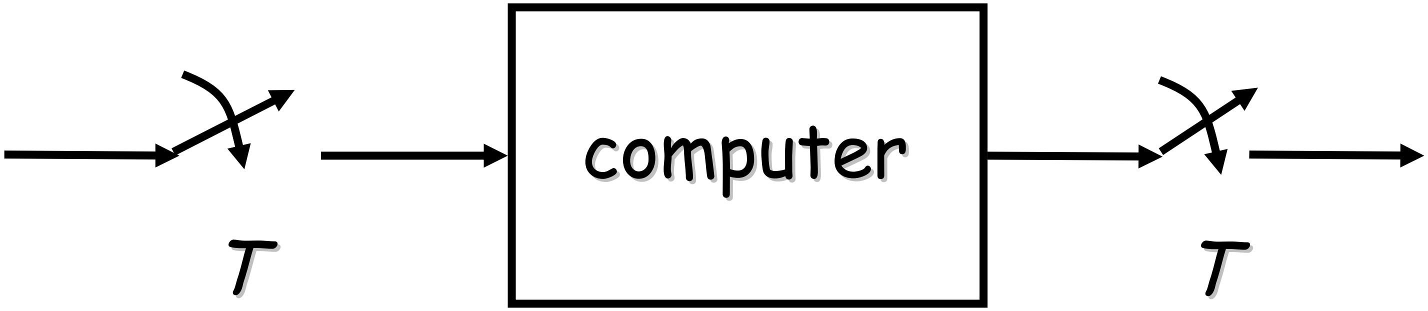
- The real world is a discrete, digital world
- Requires other process descriptions
- Design methods that take the digital nature into account

Real time

- Computer should react fast enough
- Computer should react in time

- Soft real time:
  e.g. automatic teller machine (ATM)
- Hard real time:
  all actions take place at accurately fixed time intervals

(Ideal) sampling



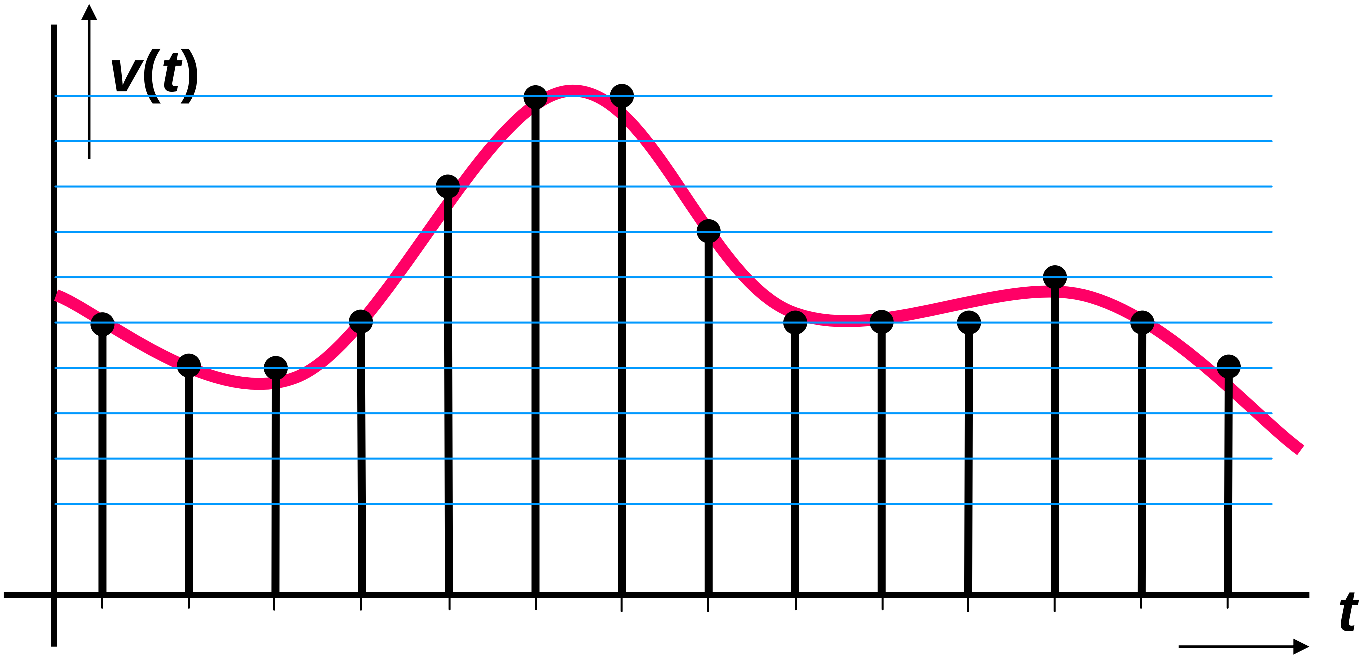
Switches close simultaneously at
$t = T, 2T, \ldots, kT, (k+1)T$

Discretisation in time

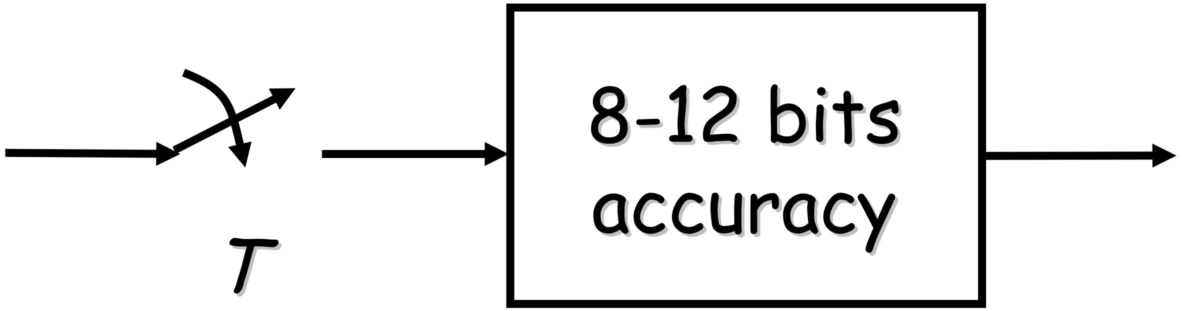
Discretisation

- Because of limited accuracy in the computer and especially in the conversion process (AD and DA-converters, encoders):
  - Discretisation in amplitude as well
  - Typically 8-12 (14, 16) bits

Sampling and Discretisation

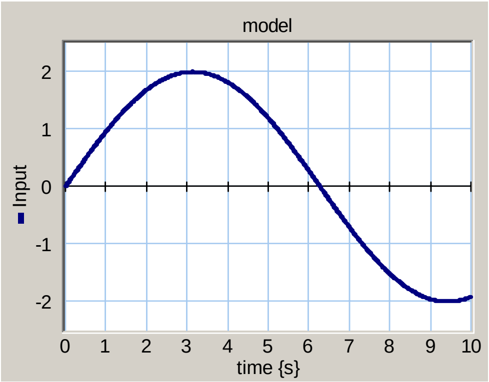


AD-conversion

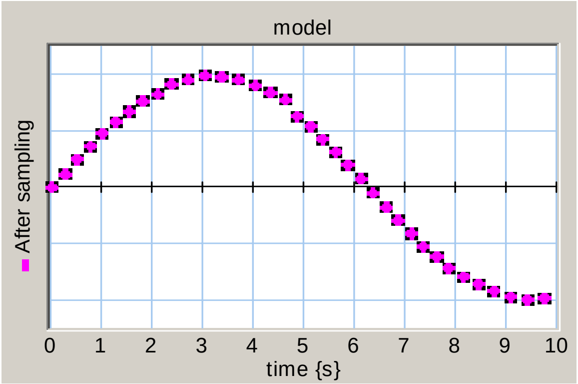


T

8-12 bits accuracy

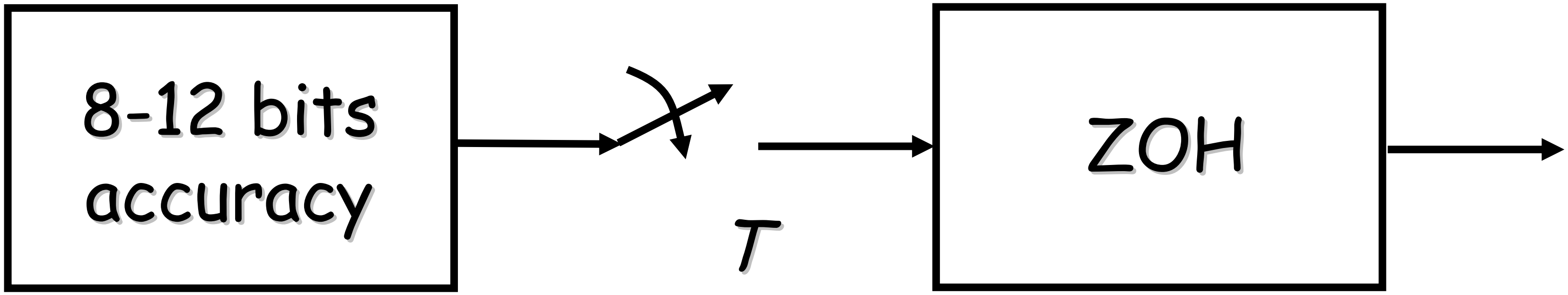


analogue
continous time



sampled and
discretised

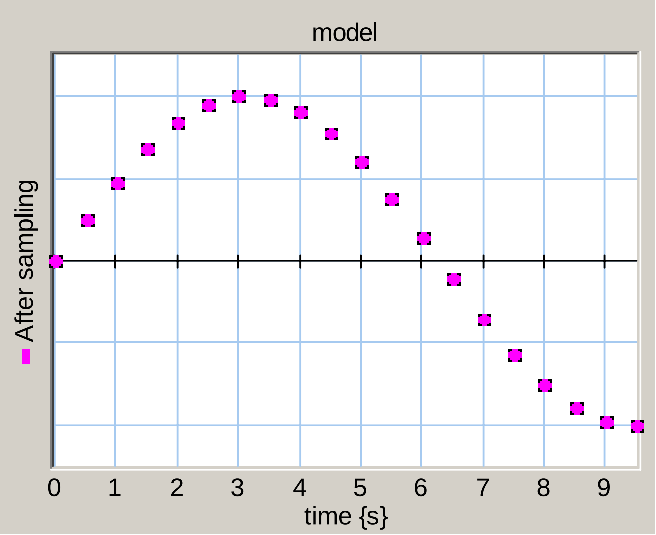
DA-conversion

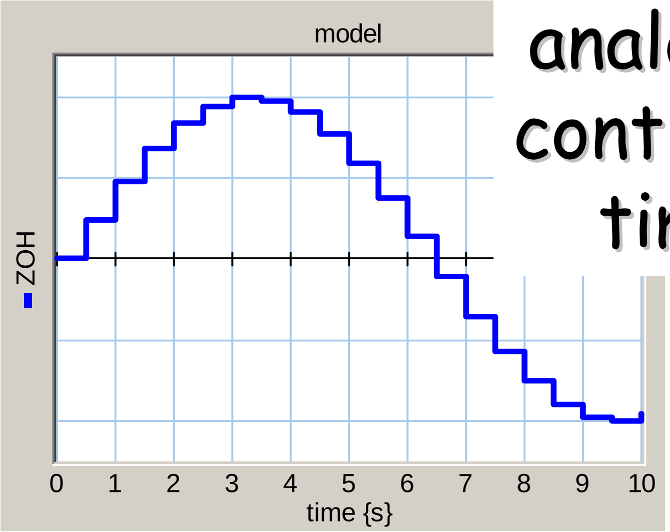


8-12 bits accuracy

T

ZOH





analogue
continous
time

Descriptions

- In a discrete system the signals have only values at the sample instants $t = T, 2T, \ldots$
- A dynamic discrete system can be described by:

$$x(k+1) = a_1 x(k) + a_2 x(k-1) + \ldots$$

$$+ b_1 u(k) + b_2 u(k-1) + \ldots$$

Difference equation

Descriptions

$$x(k+1) = a_1 x(k) + a_2 x(k-1) + \ldots$$
$$+ b_1 u(k) + b_2 u(k-1) + \ldots$$

$$x(k) = a_1 x(k-1) + a_2 x(k-2) + \ldots$$
$$+ b_1 u(k-1) + b_2 u(k-2) + \ldots$$

$$X(z) = a_1 z^{-1} X(z) + a_2 z^{-2} X(z) + \ldots$$
$$+ b_1 z^{-1} U(z) + b_2 z^{-2} U(z) + \ldots$$

$$\text{with } x\left[(k-1)T\right] \rightarrow z^{-1} X(z)$$

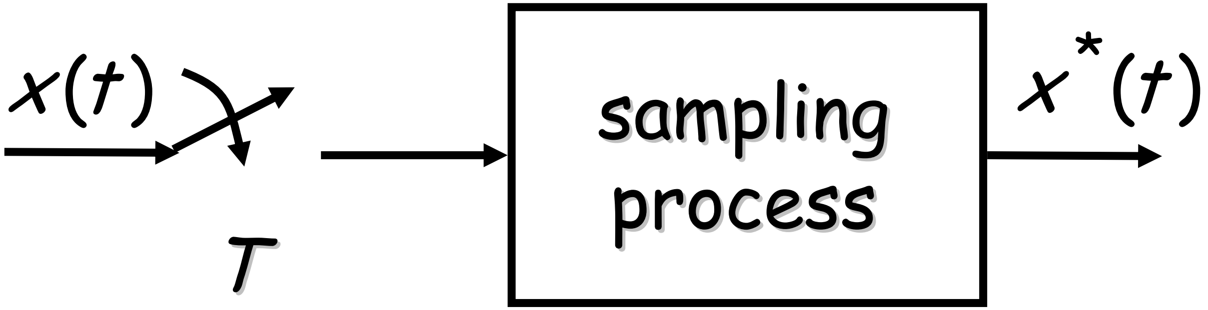
Descriptions

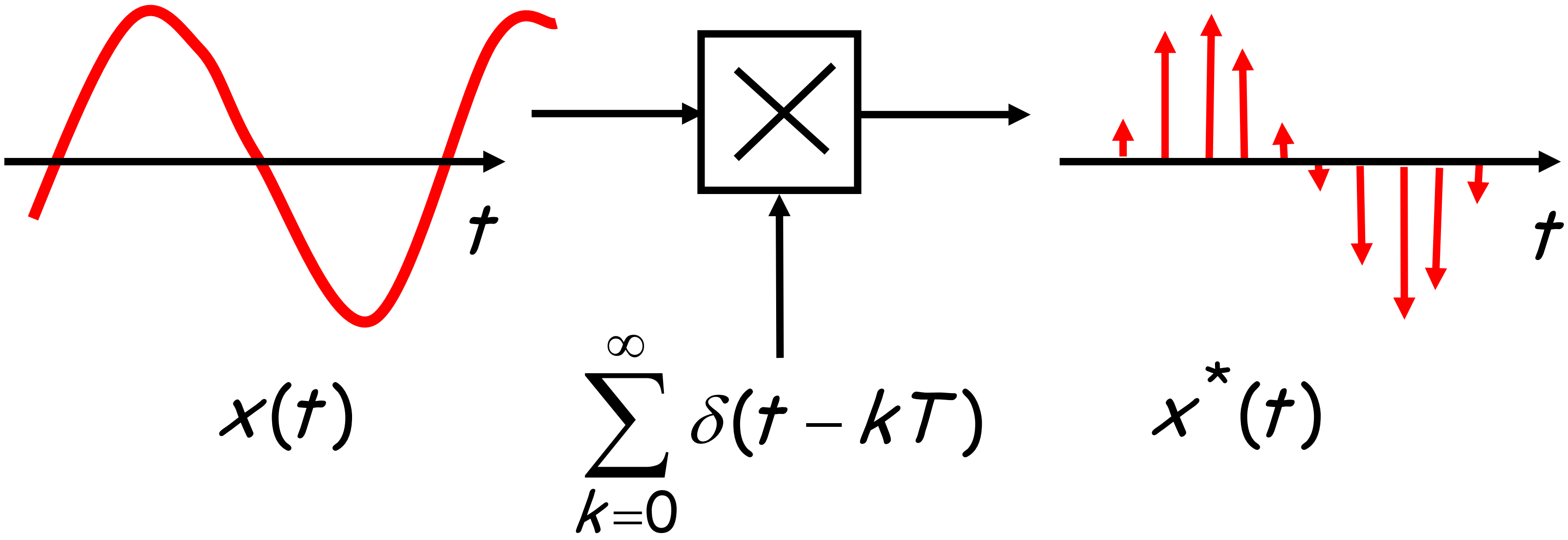
$$X(z) = a_1 z^{-1} X(z) + a_2 z^{-2} X(z) + \ldots$$
$$+ b_1 z^{-1} U(z) + b_2 z^{-2} U(z) + \ldots$$

$$\left(1 - a_1 z^{-1} - a_2 z^{-2} - \ldots\right) X(z) = \left(b_1 z^{-1} + b_2 z^{-2} + \ldots\right) U(z)$$

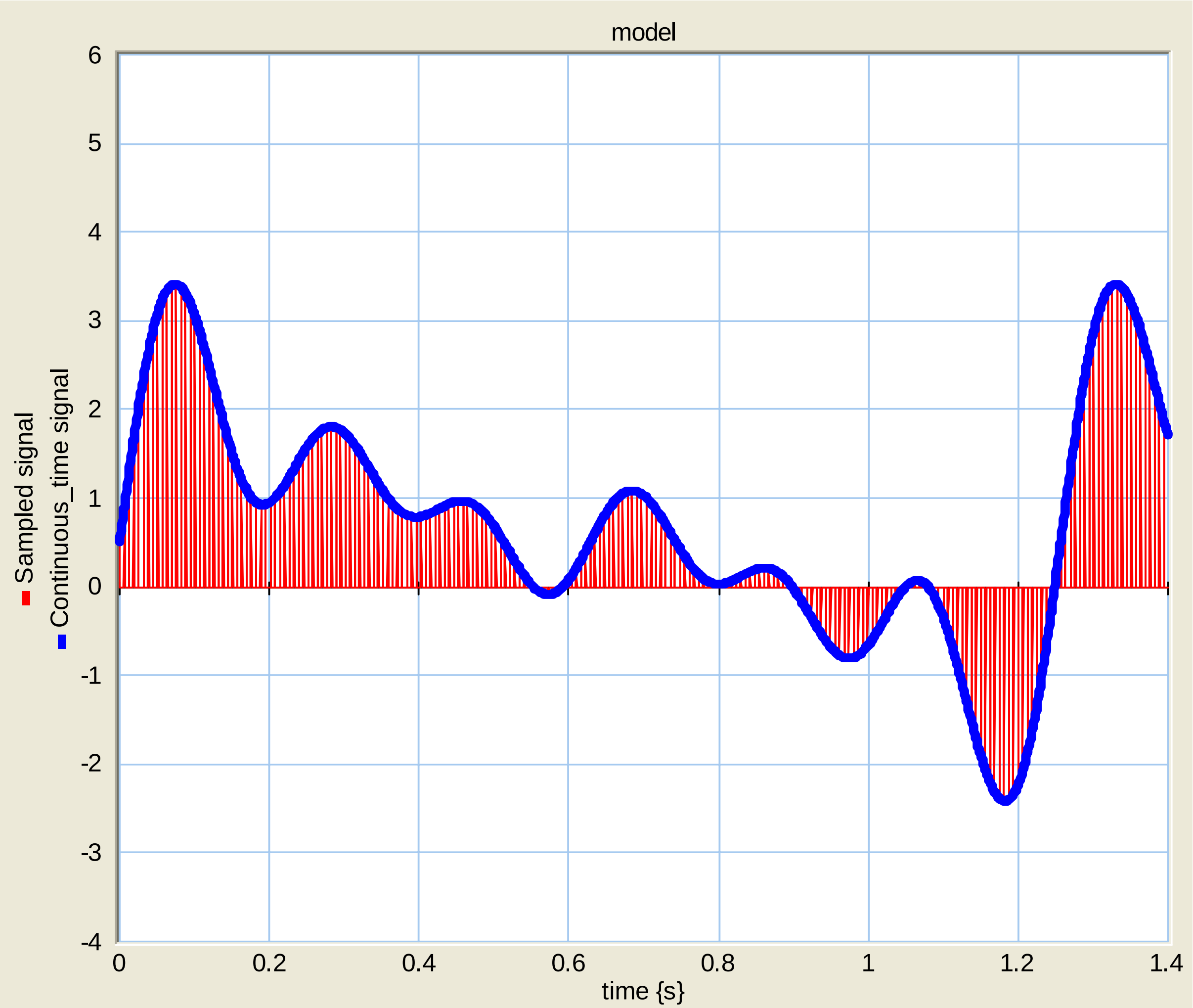
$$\frac{X(z)}{U(z)} = \frac{\left(b_1 z^{-1} + b_2 z^{-2} + \ldots\right)}{\left(1 - a_1 z^{-1} - a_2 z^{-2} - \ldots\right)}$$

Ideal sampling





Ideal sampling



20-sim

sampling_Delta

Ideal sampling

$$x^*(t) = \sum_{k=0}^{\infty} x(kT)\delta(t - kT)$$

After Laplace transformation

$$X^*(s) = \sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

Ideal sampling

$$X^*(s) = \sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

$$\omega_s = \frac{2\pi}{T} \text{ is}$$

the sampling frequency

for m integer:

$$X^*(s + jm\omega_s) = \sum_{k=0}^{\infty} x(kT)e^{-k(s+jm\omega_s)T}$$

$$X^*(s + jm\omega_s) = \sum_{k=0}^{\infty} x(kT)e^{-ksT}e^{-jkm2\pi}$$

Compare systems with dead time

Ideal sampling

$$X^*(s + jm\omega_s) = \sum_{k=0}^{\infty} x(kT)e^{-ksT}e^{-jkm2\pi}$$

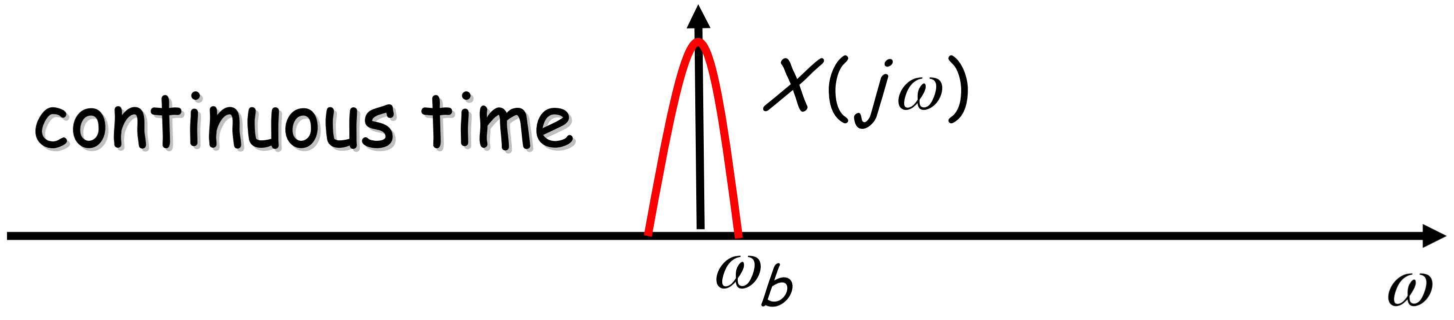
$$e^{-jkm2\pi} \ = \ 2\pi \text{ extra phase shift}$$

$$\left|e^{-jkm2\pi}\right| = 1$$

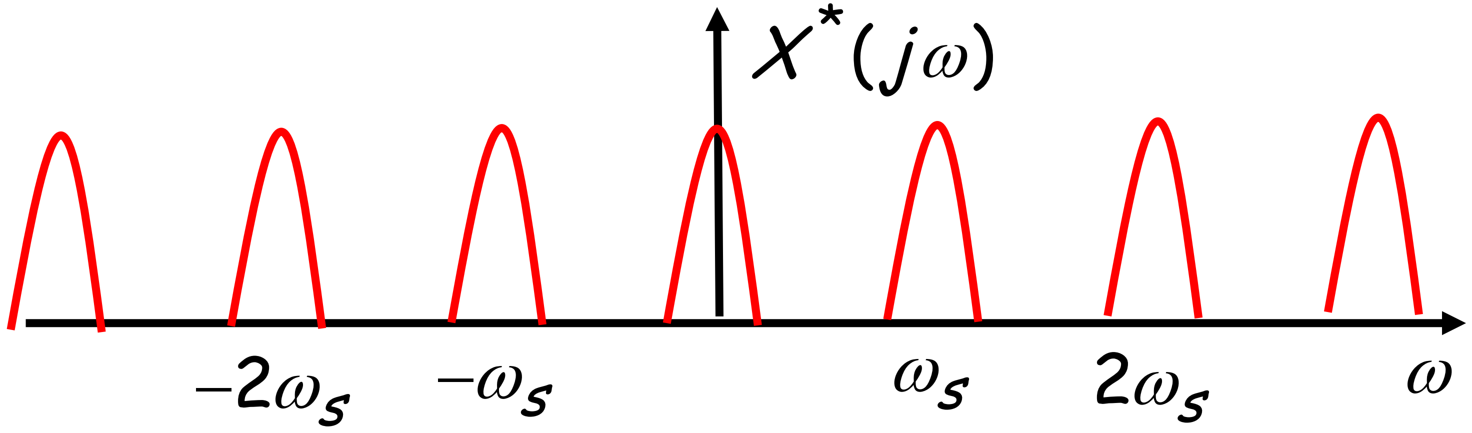
$$\left|X^*(s + jm\omega_s)\right| = \left|X^*(s)\right|$$

$$\left|X^*(j\omega + jm\omega_s)\right| = \left|X^*(j\omega)\right|$$

Frequency domain



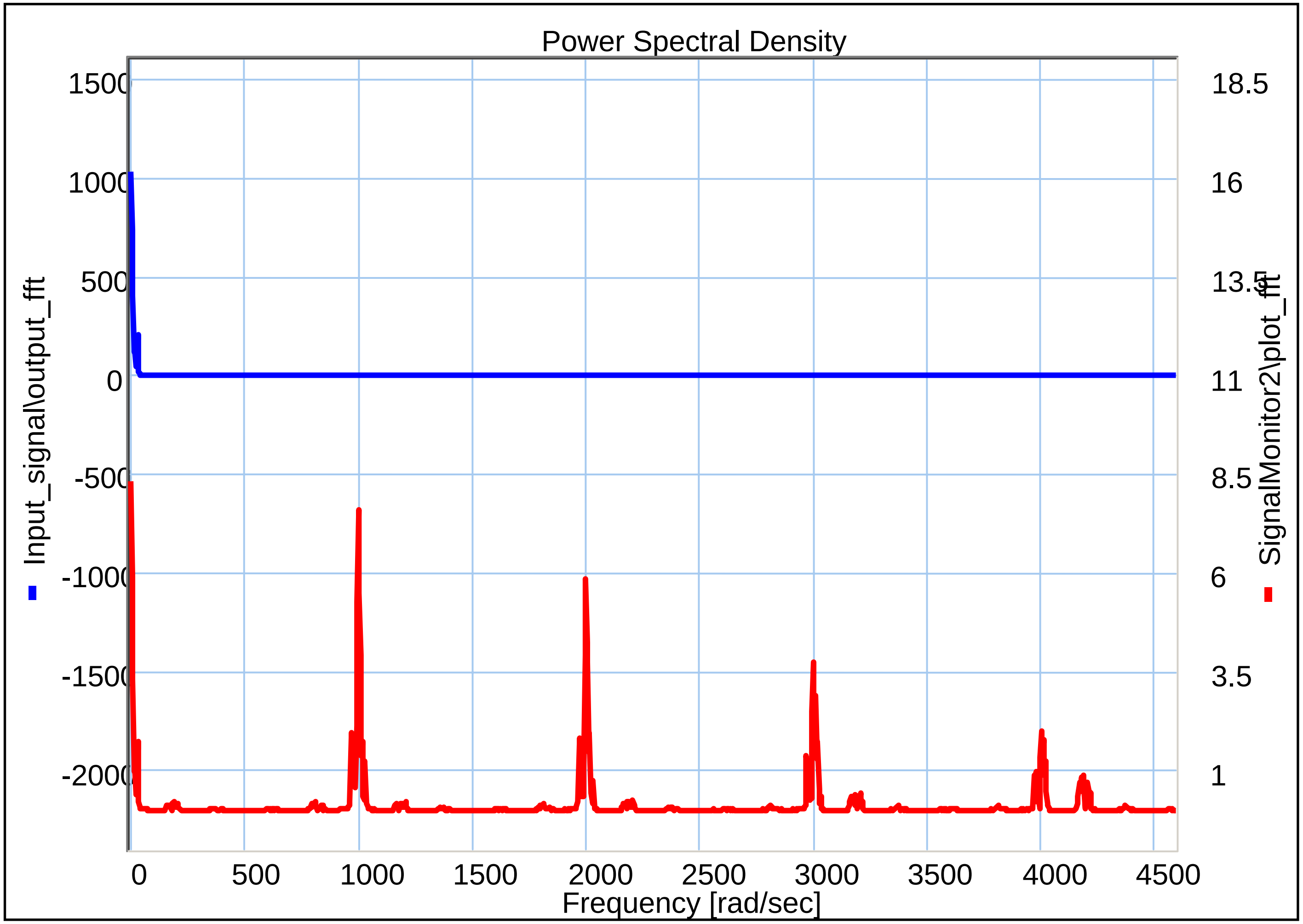
$$\left|X^*(j\omega + jm\omega_s)\right| = \left|X^*(j\omega)\right|$$



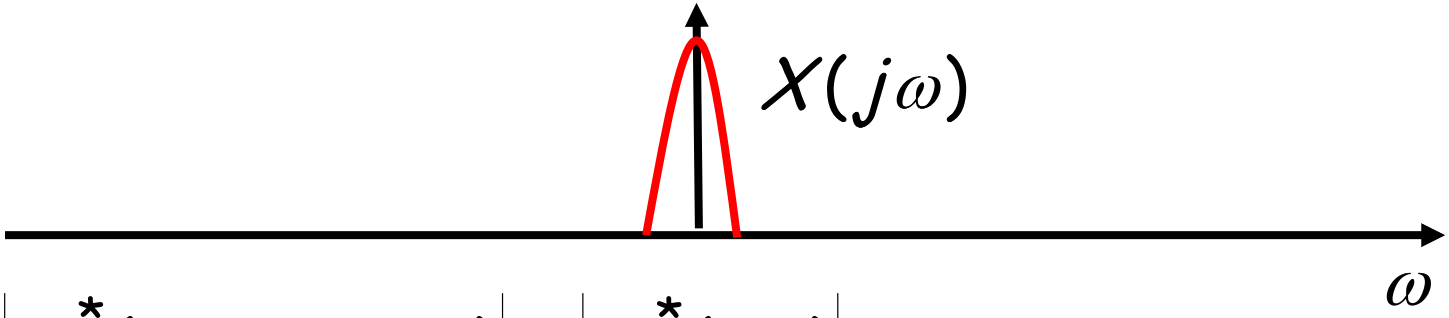
Frequency domain

20-sim

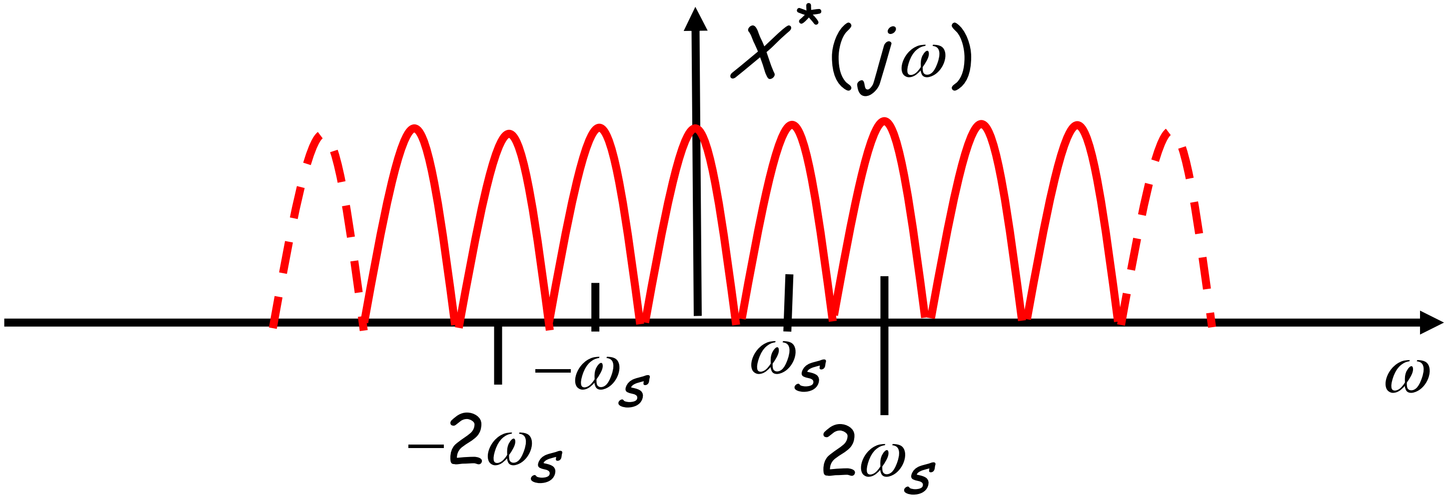
sampling_Delta



Frequency domain



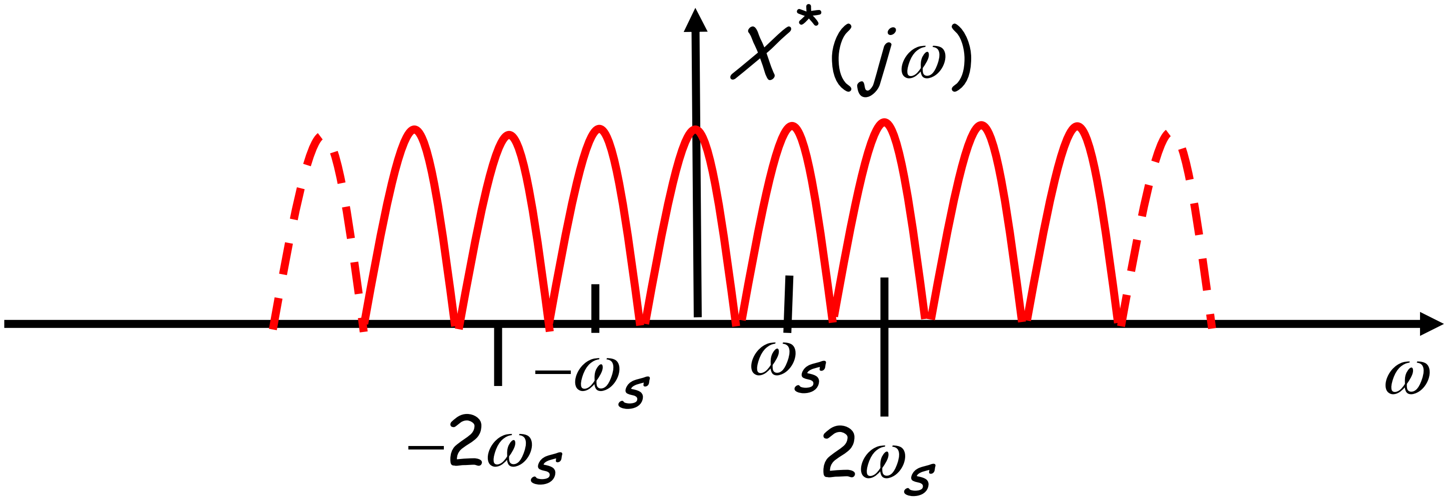
$$\left|X^*(j\omega + jm\omega_s)\right| = \left|X^*(j\omega)\right|$$



Frequency domain

Nyquist frequency: $\omega_s = 2\omega_b$

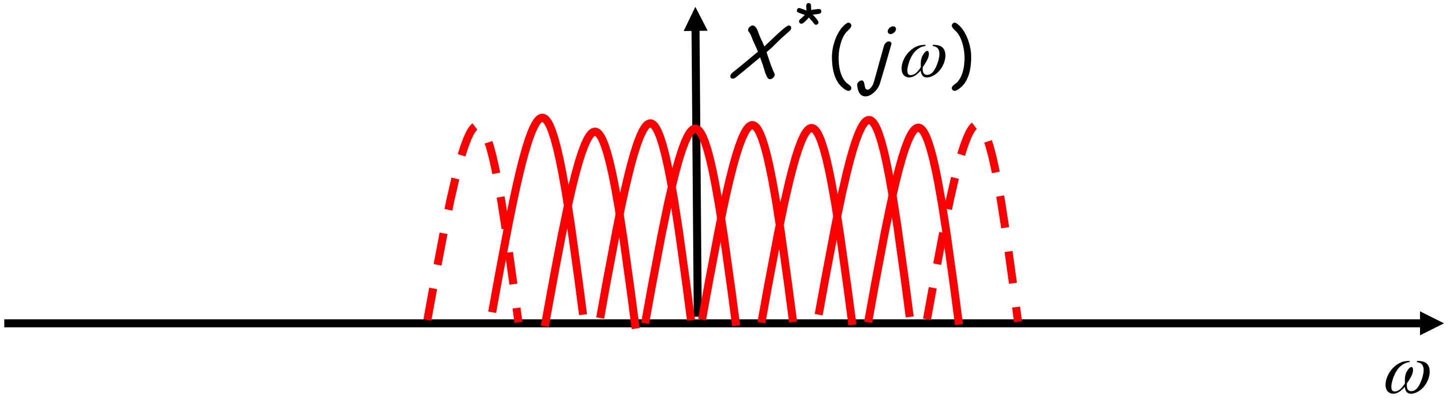
Theoretically (!) possible to reconstruct the signal



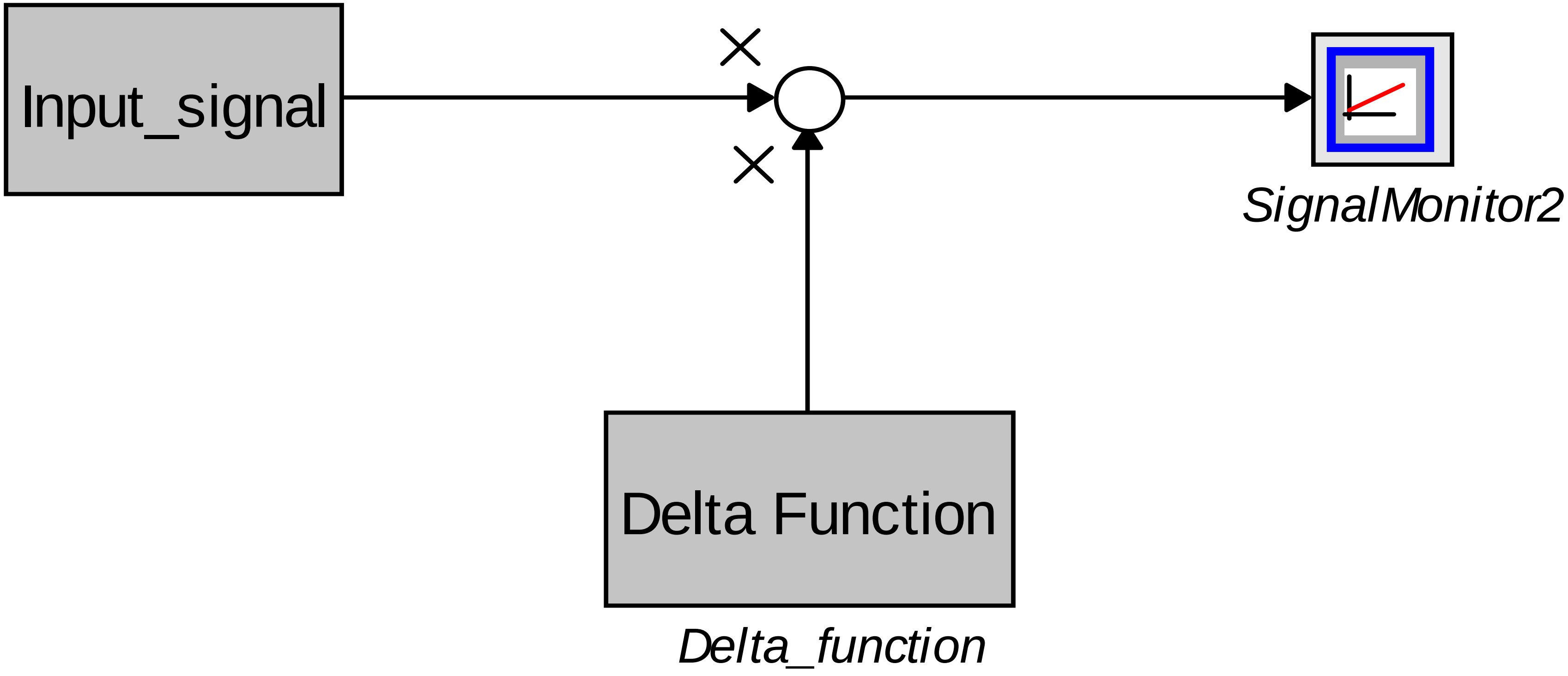
Frequency domain

If $\omega_s < 2\omega_b$ reconstruction not possible anymore

ALIASING



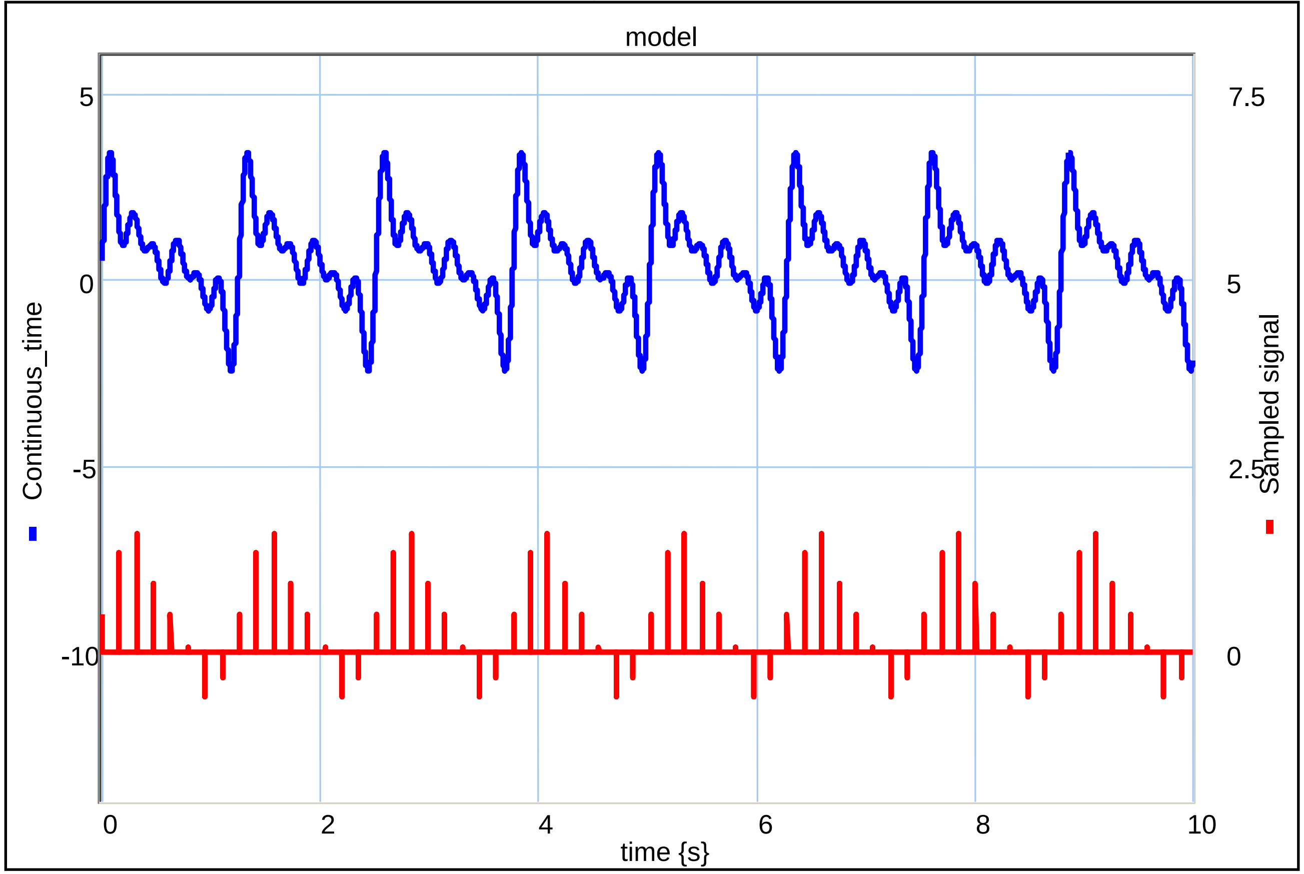
Reconstruction



Reconstruction

If $\omega_s < 2\omega_b$ construction not possible anymore

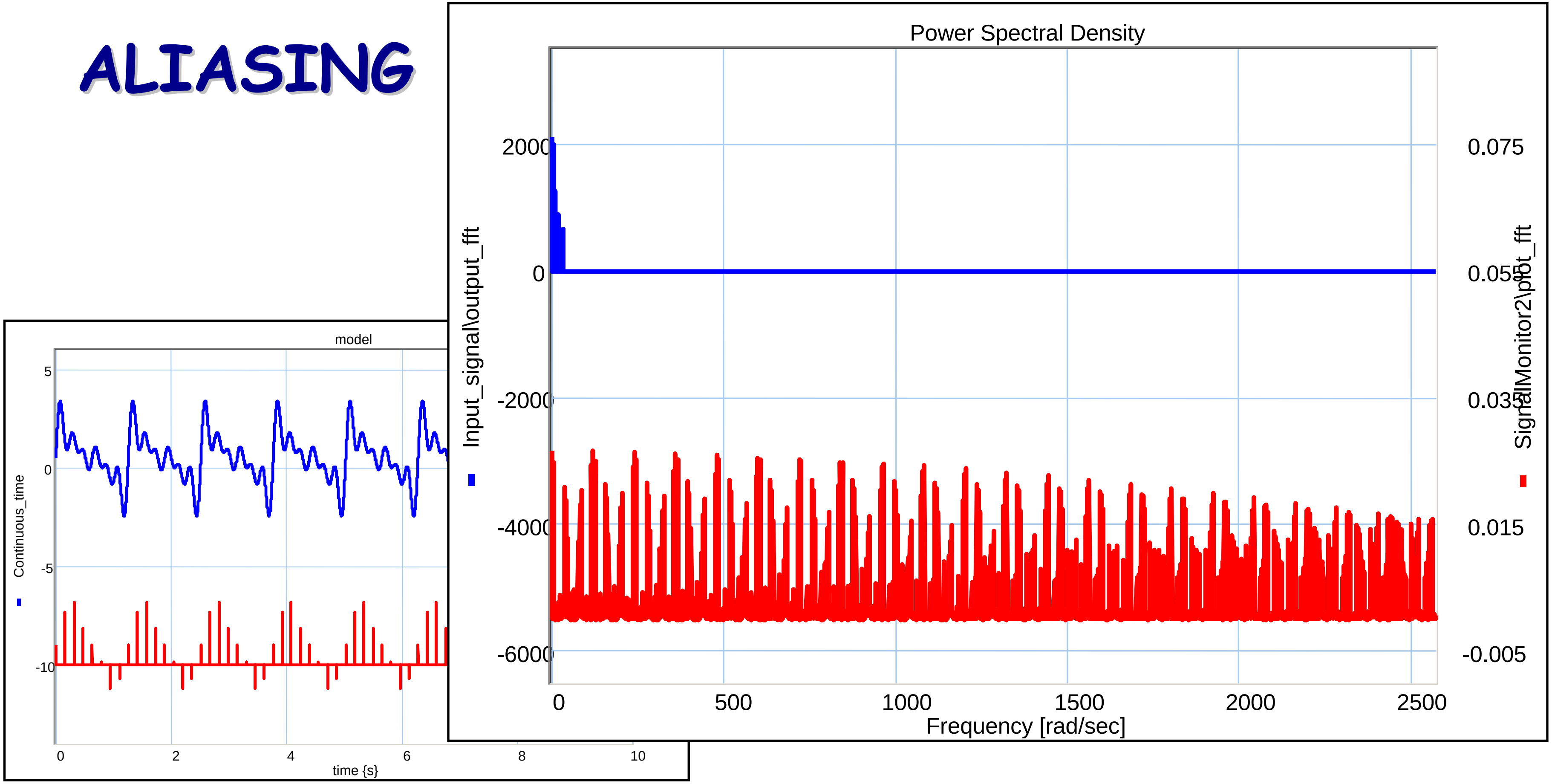
ALIASING



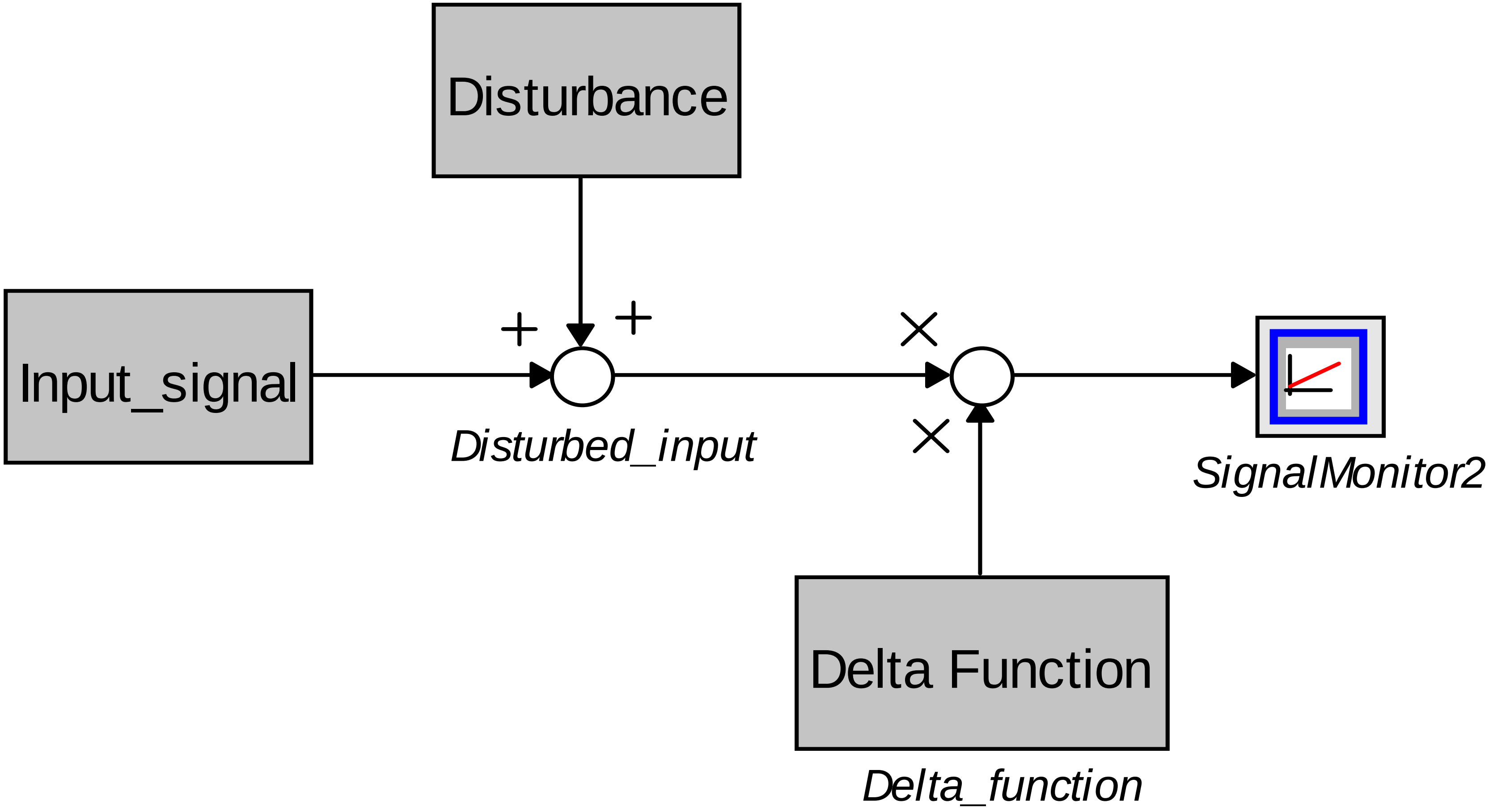
Reconstruction

If $\omega_s < 2\omega_b$ construction not possible anymore

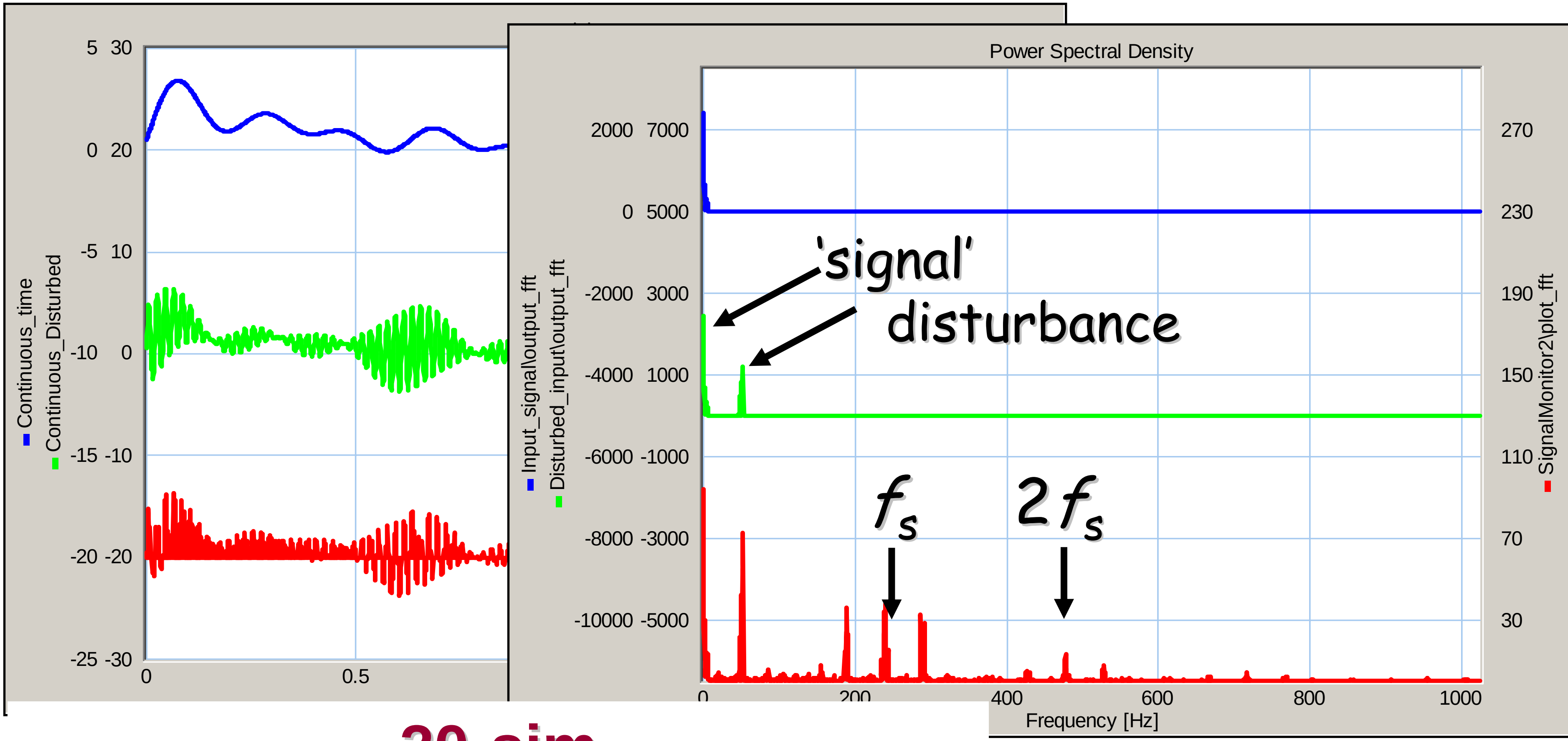
ALIASING



Reconstruction



Sampling f_s=250 Hz (1500 rad/s)



20-sim

sampling_Delta with disturbance

Sampling f_s=50 Hz (300 rad/s)

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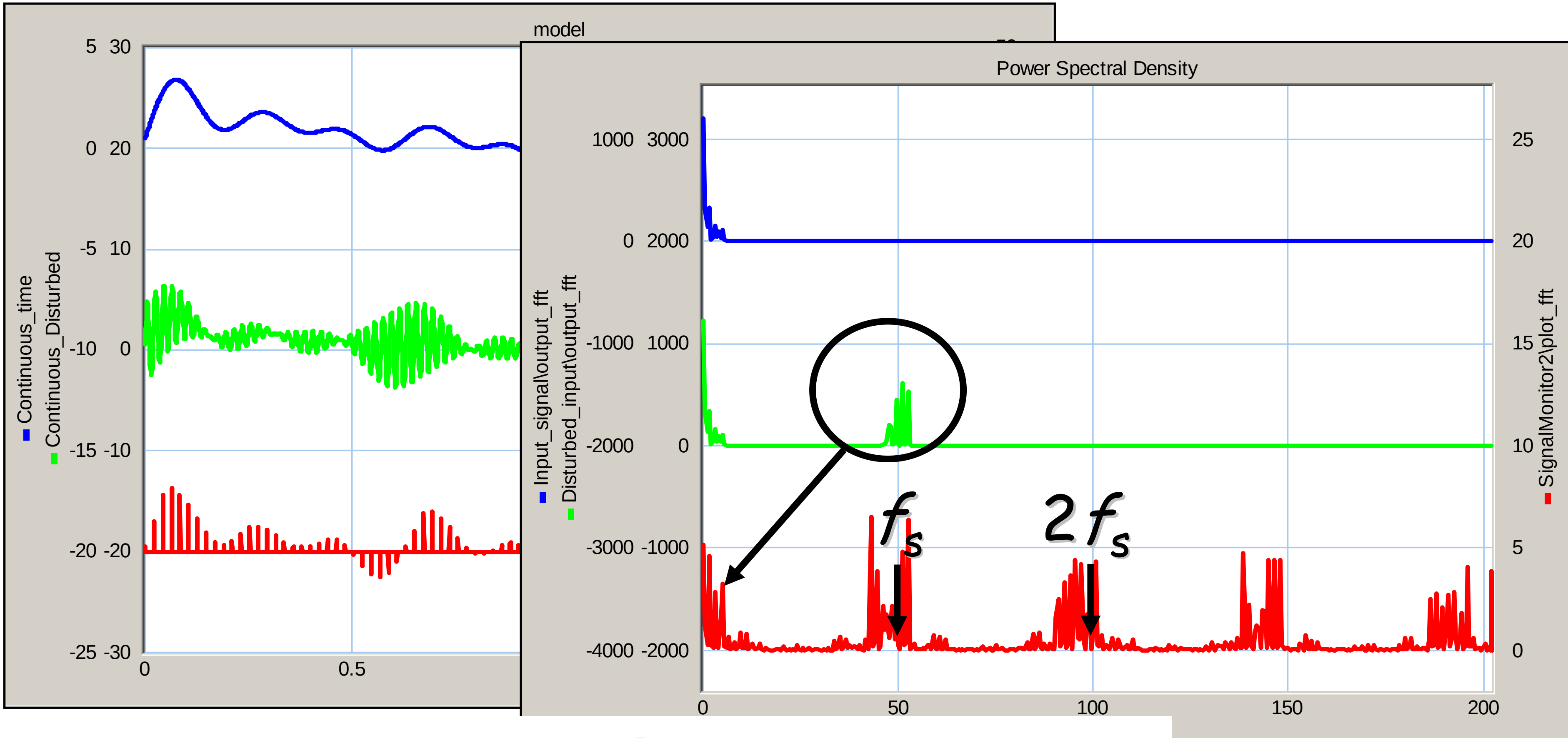

20-sim

sampling_Delta with disturbance_300

Sampling f_s=20 Hz (120 rad/s)

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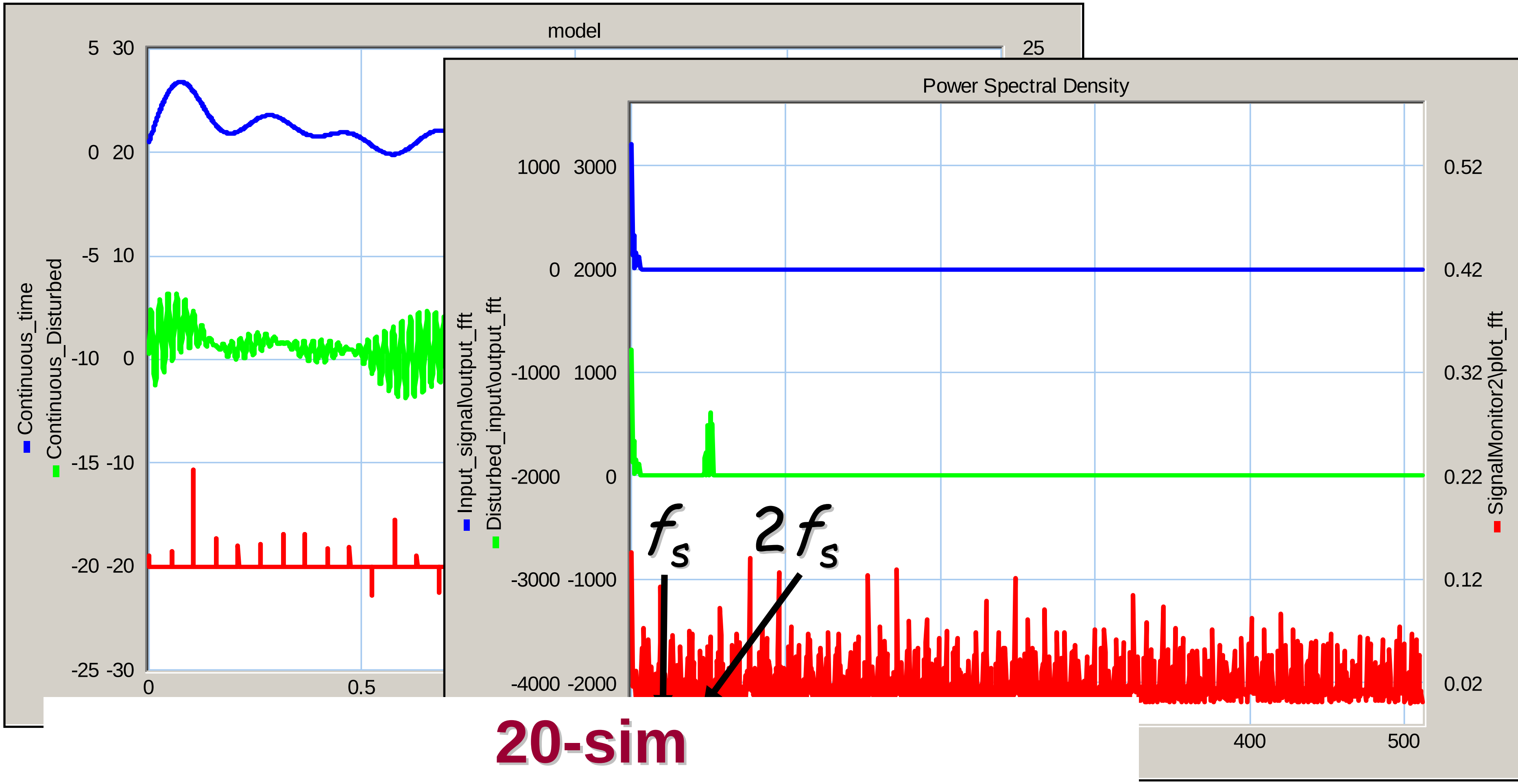
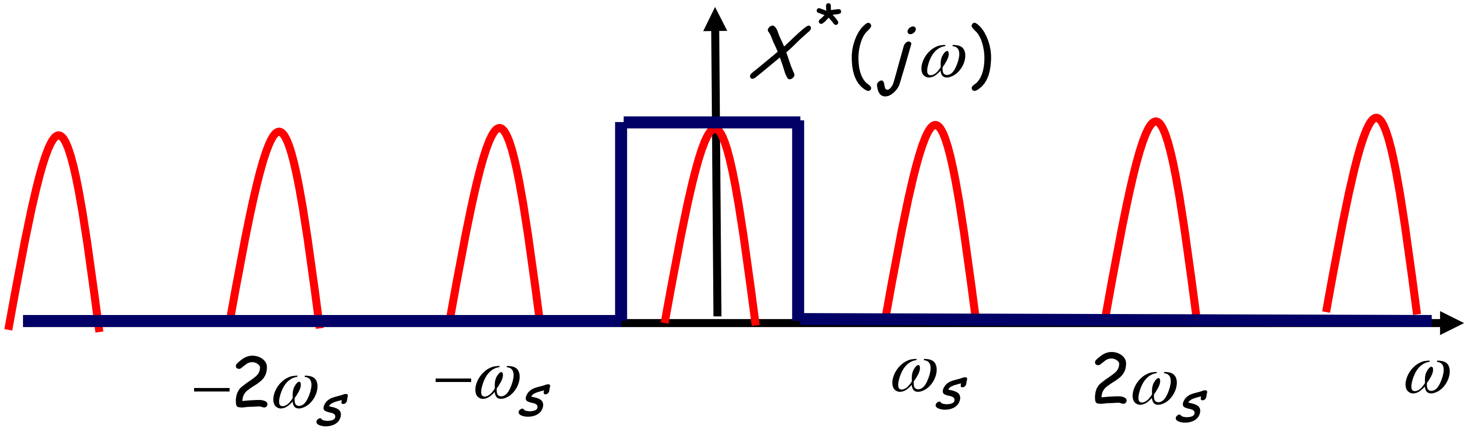


20-sim

sampling_Delta with disturbance_120

Reconstruction

We would like (?) to have the following filter



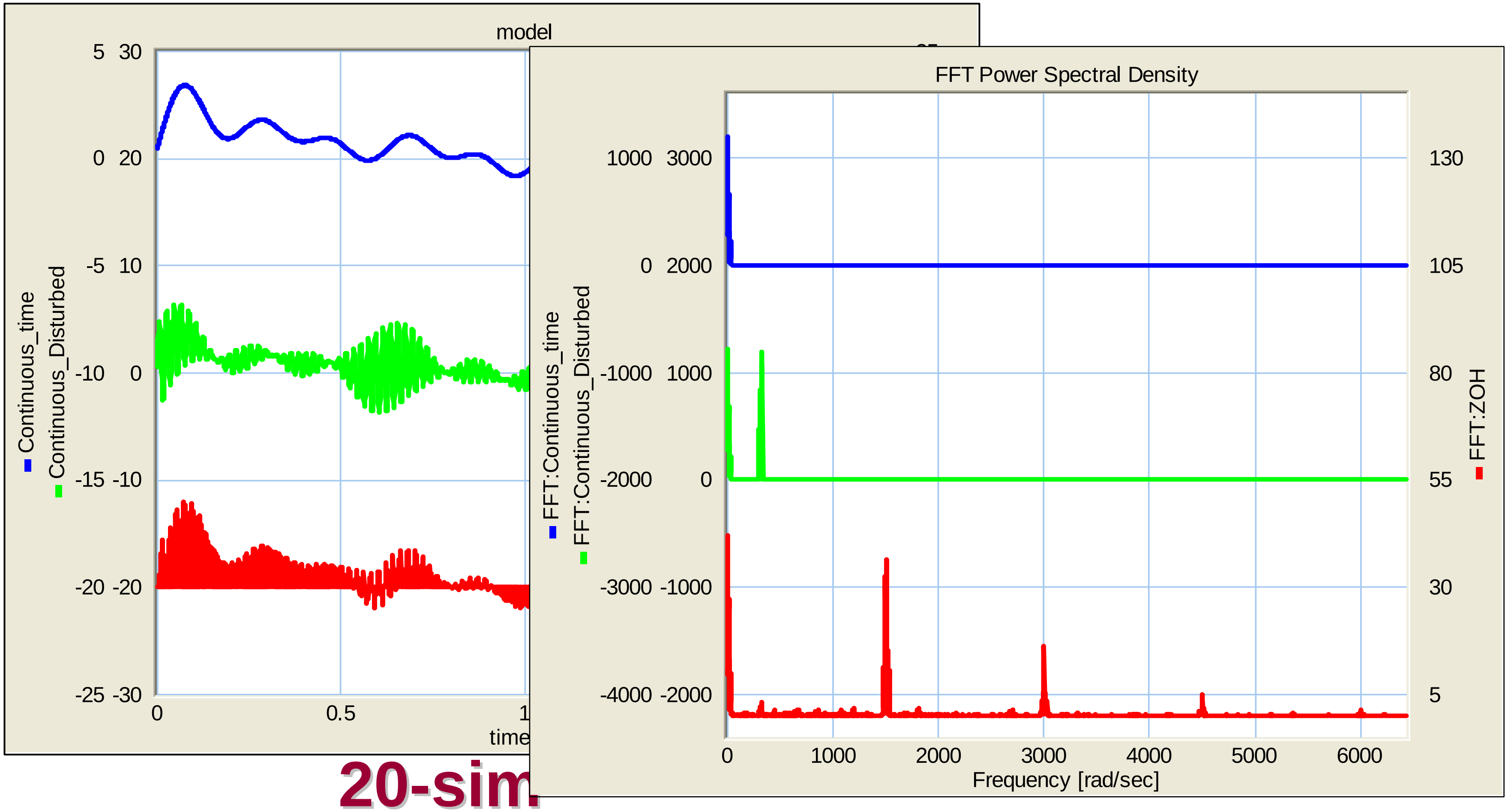
In Practice

- Use an anti aliasing (low-pass) filter at the input
- Take care that $\omega_s >> 2\ \omega_{n,max}$ *e.g.*

  $\omega_s = 10\ \omega_{n,max}$

  Where $\omega_{n,max}$ is the distance from the 'fastest' poles to the origin

Anti Aliasing ω_s=1500 rad/s



20-sim

sampling_Delta _anti_aliasing

Anti Aliasing ω_s=150 rad/s

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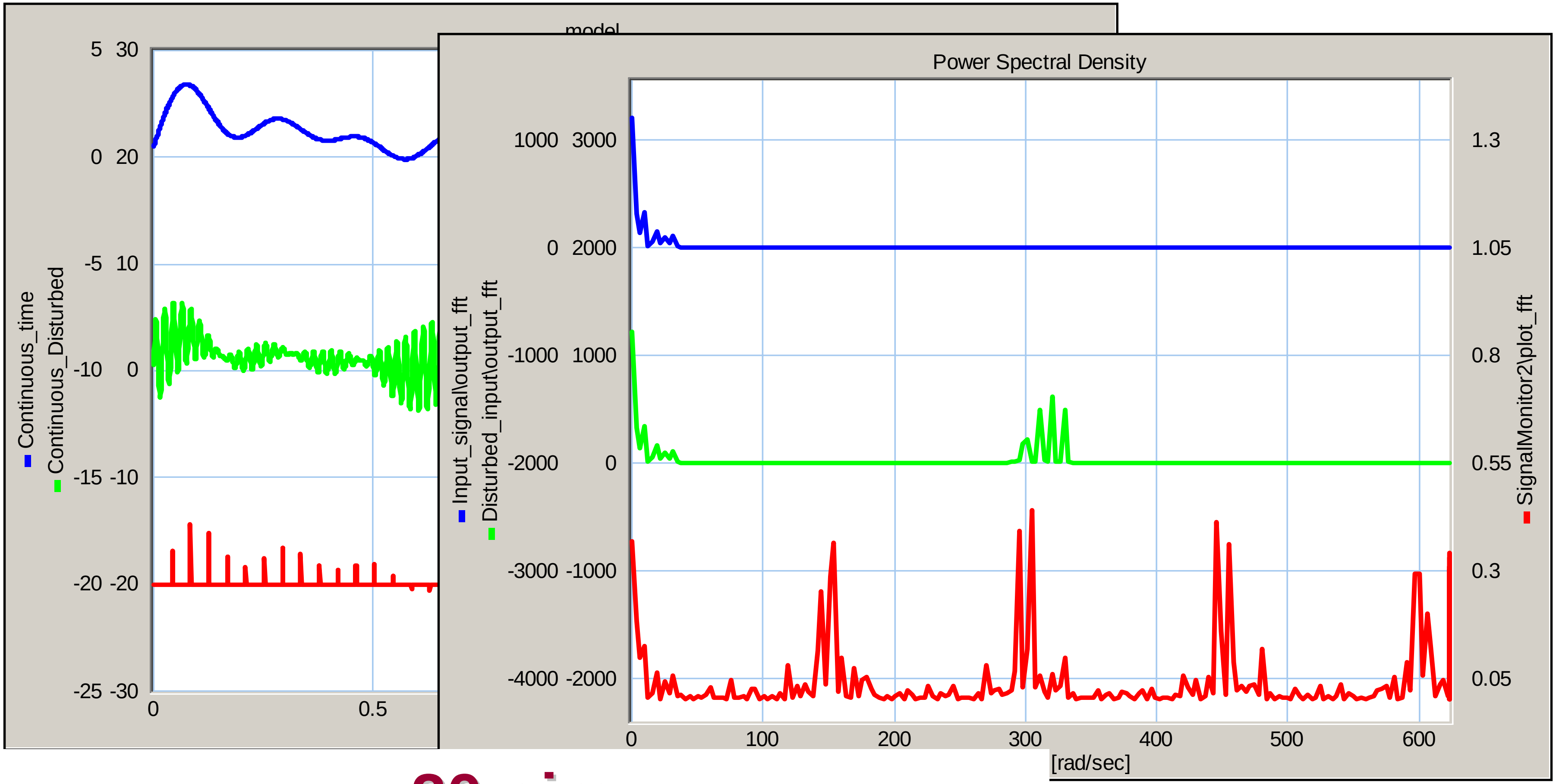


20-sim

sampling_Delta _anti_aliasing_150

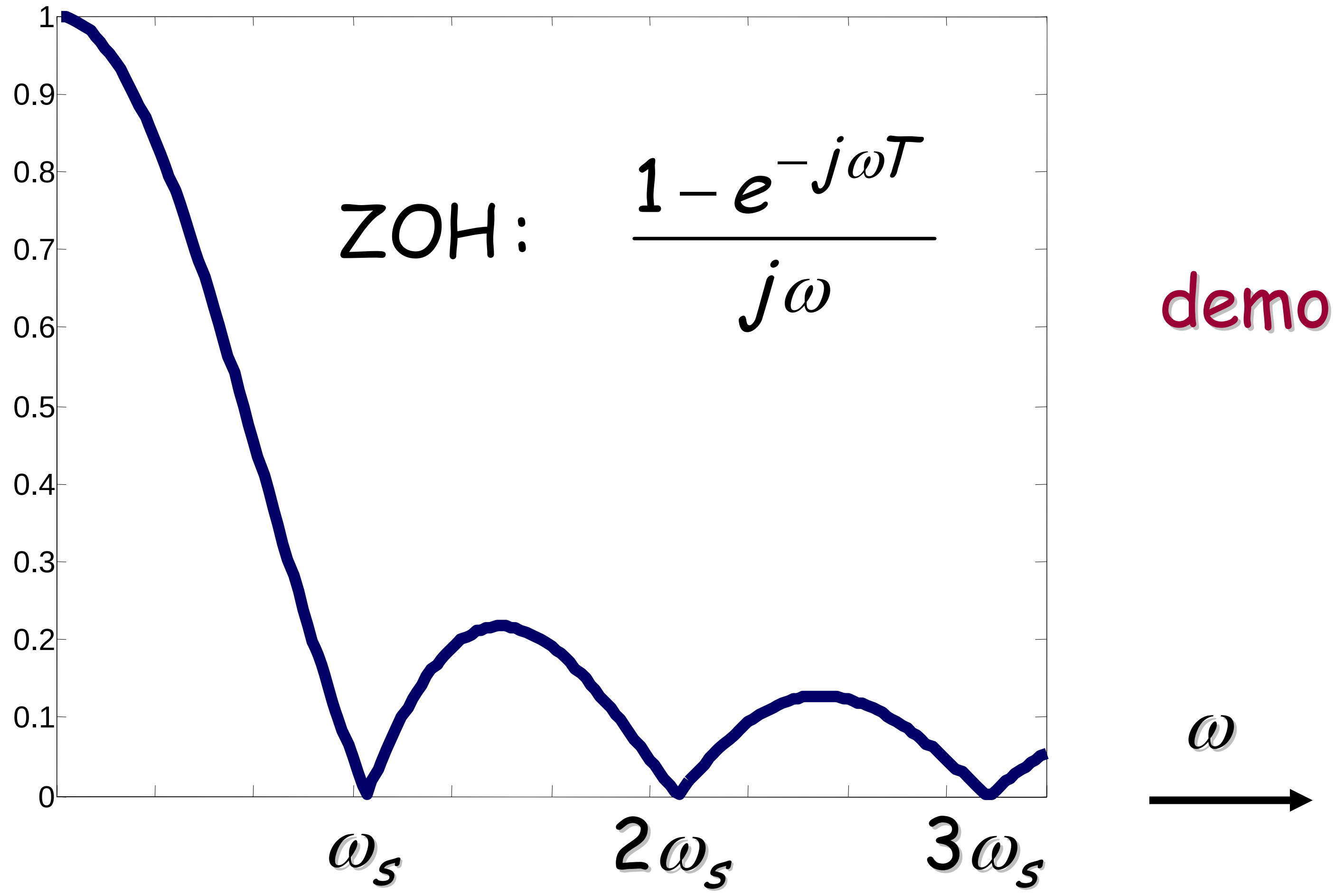
Reconstruction: ZOH

$$\text{ZOH}: \quad \frac{1-e^{-j\omega T}}{j\omega}$$

MATLAB:

```
for k = 1:1:200,
w(k)=k/10
x(k)=abs((1-exp(-j*w(k)))/w(k))
end
plot (w,x)
```

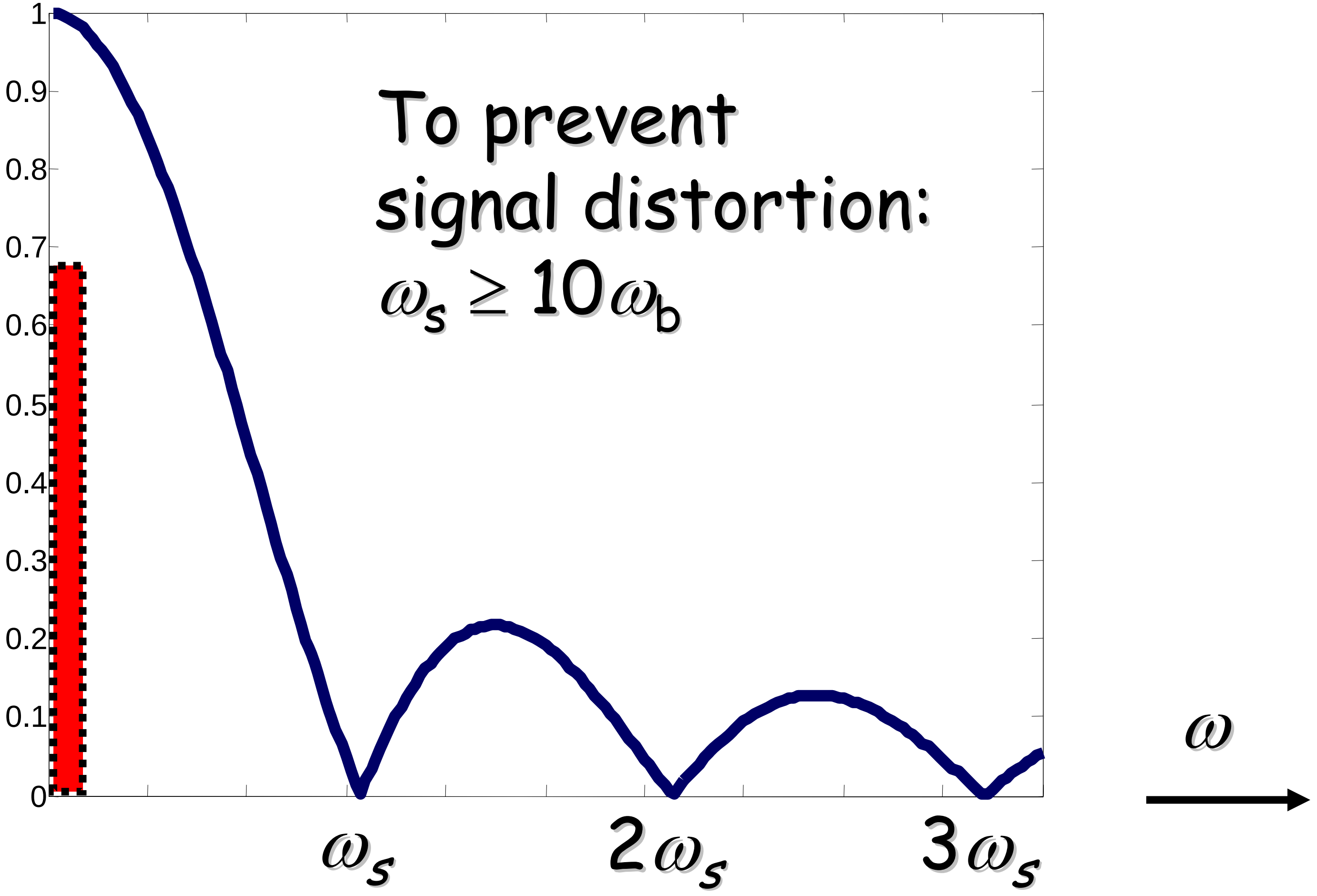
Reconstruction: ZOH



ZOH: $\dfrac{1-e^{-j\omega T}}{j\omega}$

demo

Reconstruction: ZOH



To prevent signal distortion:

$\omega_s \geq 10\omega_b$

In Practice

- Use an anti aliasing (low-pass) filter at the input
- Take care that $\omega_s >> 2\ \omega_b$ e.g. $\omega_s = 10\ \omega_b$

- To eliminate high frequencies at output:
- Use a low pass filter

Output filtering

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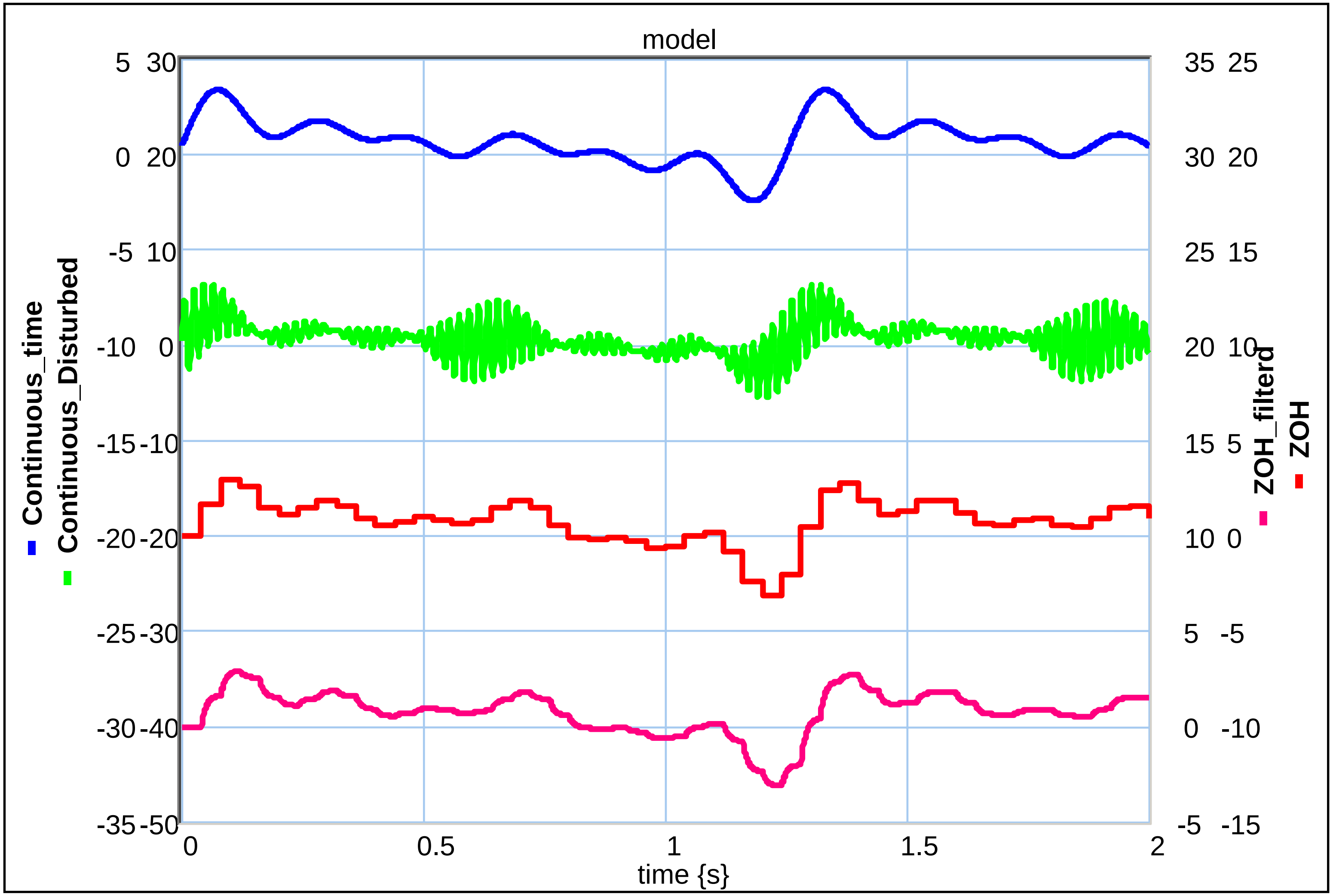


Descriptions

time domain

z-domain

Continuous time system seen from the computer (1^{st} order)

Consider the first order system: $\frac{X(s)}{U(s)}=\frac{Ka}{s+a}$

$$x(t)=e^{-at}x(0)+K'\int_{0}^{T}e^{-a(t-\tau)}u(\tau)d\tau$$

In a discrete system with ZOH:

$$x[(k+1)T]=e^{-aT}x(kT)+K'\left[\int_{0}^{T}e^{-at}dt\right]u(kT)$$

Continuous time system seen from the computer

$$x[(k+1)T] = e^{-aT}x(kT) + K'\left[\int_0^T e^{-at}dt\right]u(kT)$$

$$x[(k+1)T] = e^{-aT}x(kT) - \frac{Ka}{a}\left[e^{-at}\right]\Big|_0^T u(kT)$$

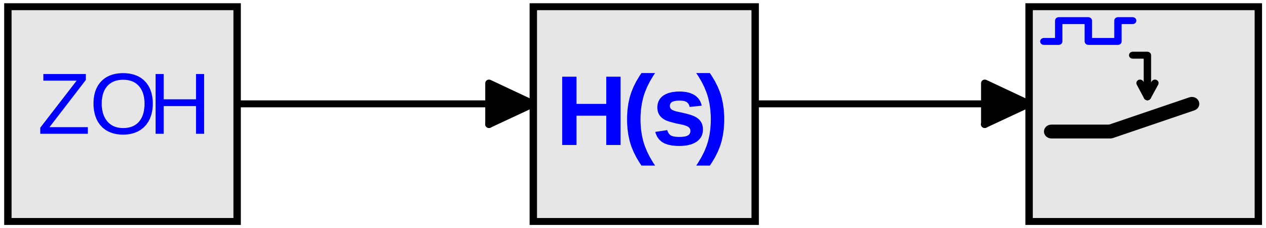
$$x[(k+1)T] = e^{-aT}x(kT) + K[1-e^{-aT}]u(kT)$$

$$zX(z) = e^{-aT}X(z) + K[1-e^{-aT}]U(z)$$

$$\frac{X(z)}{U(z)} = \frac{K[1-e^{-aT}]}{z-e^{-aT}}$$

Exact description in z

- In general: solving $x\{(k+1)T\} = f\{x(kT\}, u(kT)\}$ yields an exact description in z of the continuous time system (with ZOH), as seen by the computer:



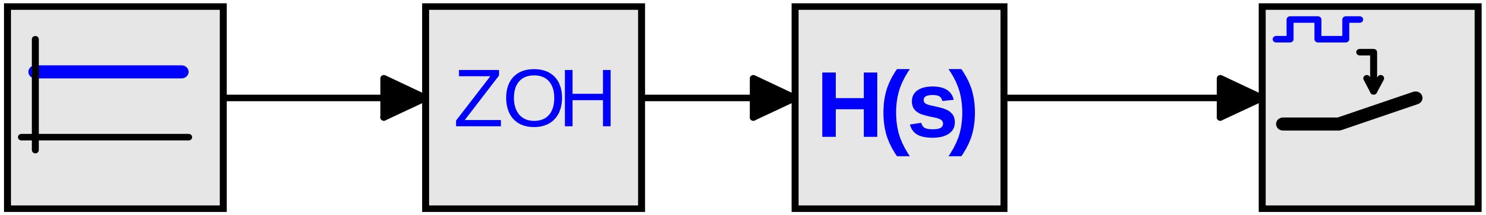
Find exact solution

- Describe $H(s)$ in state space form
- When a zero-order hold is used, the solution of the given system is:

$$x[(k+1)T] = e^{AT} x(kT) + \left[\int_0^T e^{At} dt \right] bu(kT)$$

- Coupled first-order systems !!

- One way to determine the various matrices is to simulate:



Homogeneous solution

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$$x[(k+1)T] = e^{-AT}x(kT) + \xcancel{\left[\int_0^T e^{At}dt\right]bu(kT)}$$

$$\left.\begin{aligned} x[(k+1)T] &= e^{-AT}x(kT) \\ x[(k+1)T] &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} x(kT) \end{aligned}\right\} \text{if } u = 0$$

e.g.
2nd
order

$$\left.\begin{aligned} a_{11} &= x_1\left[(k+1)T\right] \\ a_{21} &= x_2\left[(k+1)T\right] \end{aligned}\right\} \text{for } x_1[kT] = 1 \text{ and } x_2[kT] = 0$$

$$\left.\begin{aligned} a_{12} &= x_1\left[(k+1)T\right] \\ a_{22} &= x_2\left[(k+1)T\right] \end{aligned}\right\} \text{for } x_1[kT] = 0 \text{ and } x_2[kT] = 1$$

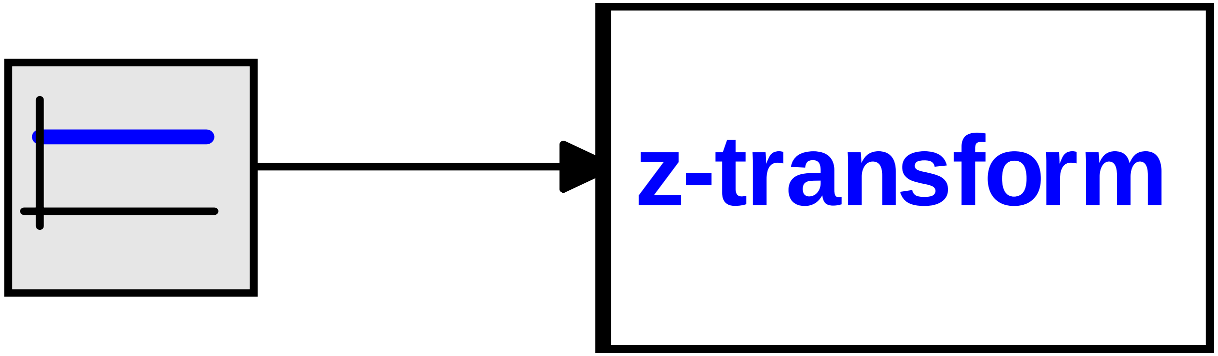
Particular solution

$$x[(k+1)T] = e\xcancel{^{-AT}x}(kT) + \left[\int_0^T e^{At}dt\right] bu(kT)$$

$$\left. x[(k+1)T] = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} u(kT) \right\} \text{ if } \mathbf{x}(kT) = 0$$

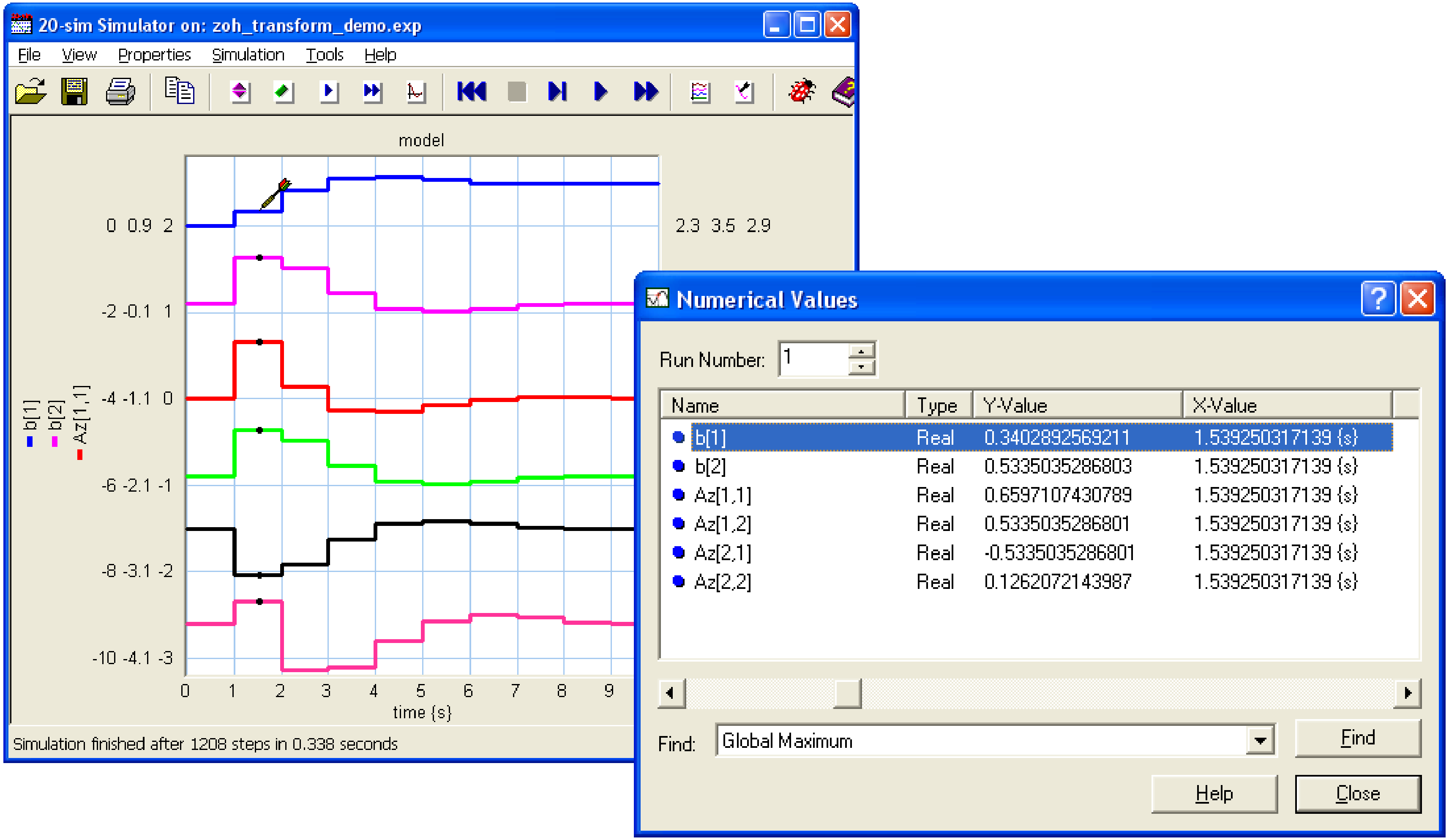
$$\left. \begin{array}{l} b_1 = x_1\left[(k+1)T\right] \\ b_2 = x_1\left[(k+1)T\right] \end{array} \right\} \text{ for } u\left[kT\right] = 1 \text{ and } \mathbf{x}\left[kT\right] = 0$$

Example 2nd-order system



20-sim demo

Example 2nd-order system



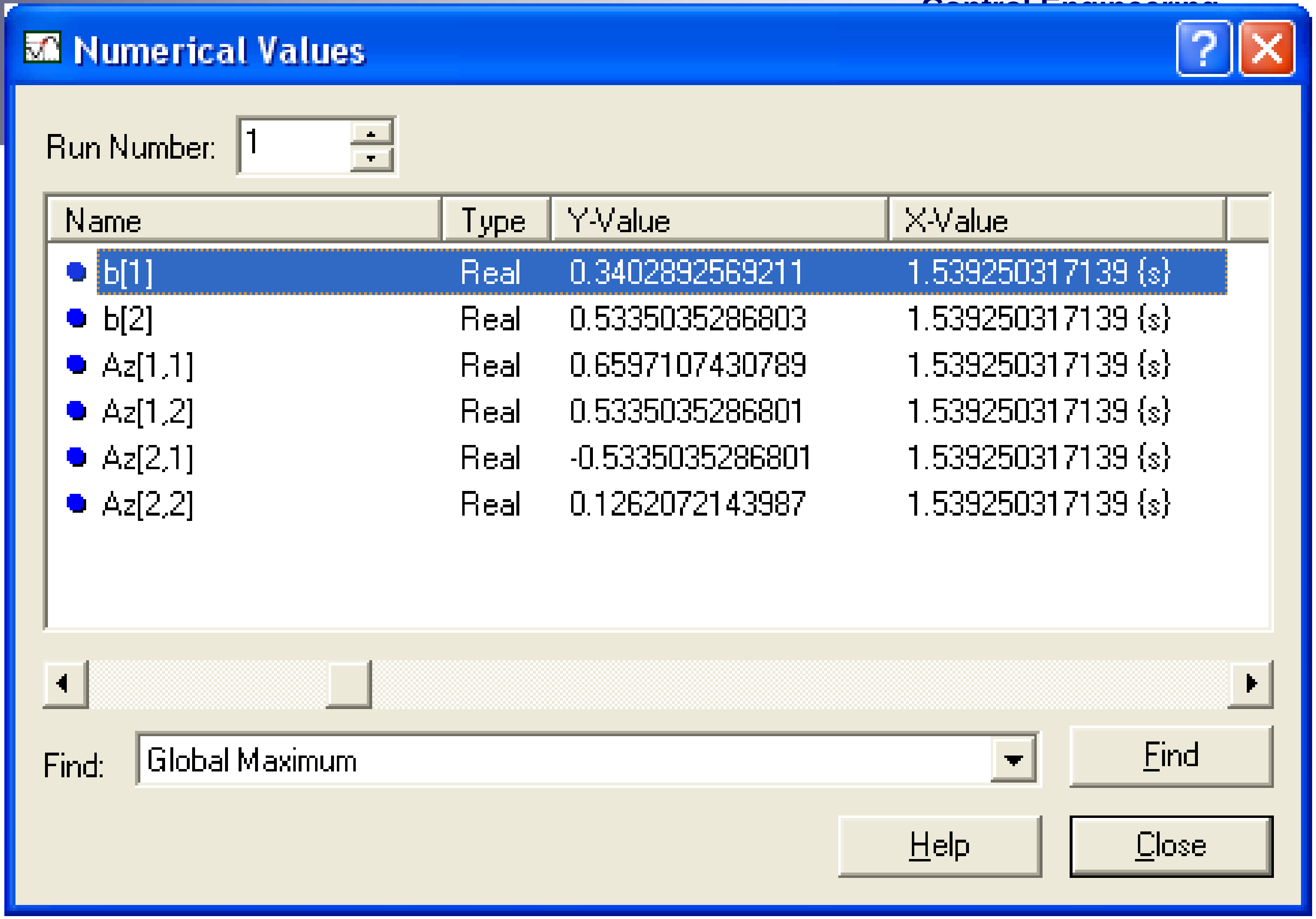
Simulation finished after 1208 steps in 0.338 seconds

Run Number: 1

Name	Type	Y-Value	X-Value
● b[1]	Real	0.3402892569211	1.539250317139 {s}
● b[2]	Real	0.5335035286803	1.539250317139 {s}
● Az[1,1]	Real	0.6597107430789	1.539250317139 {s}
● Az[1,2]	Real	0.5335035286801	1.539250317139 {s}
● Az[2,1]	Real	-0.5335035286801	1.539250317139 {s}
● Az[2,2]	Real	0.1262072143987	1.539250317139 {s}

Find: Global Maximum

Result



Run Number: 1

Name	Type	Y-Value	X-Value
b[1]	Real	0.3402892569211	1.539250317139 {s}
b[2]	Real	0.5335035286803	1.539250317139 {s}
Az[1,1]	Real	0.6597107430789	1.539250317139 {s}
Az[1,2]	Real	0.5335035286801	1.539250317139 {s}
Az[2,1]	Real	-0.5335035286801	1.539250317139 {s}
Az[2,2]	Real	0.1262072143987	1.539250317139 {s}

Find: Global Maximum

$$x[k+1]=\begin{pmatrix} 0.6597 & 0.5335 \\ -0.5335 & 0.1262 \end{pmatrix} x[k]+\begin{pmatrix} 0.3403 \\ 0.5335 \end{pmatrix} u(k)$$

Z-transform

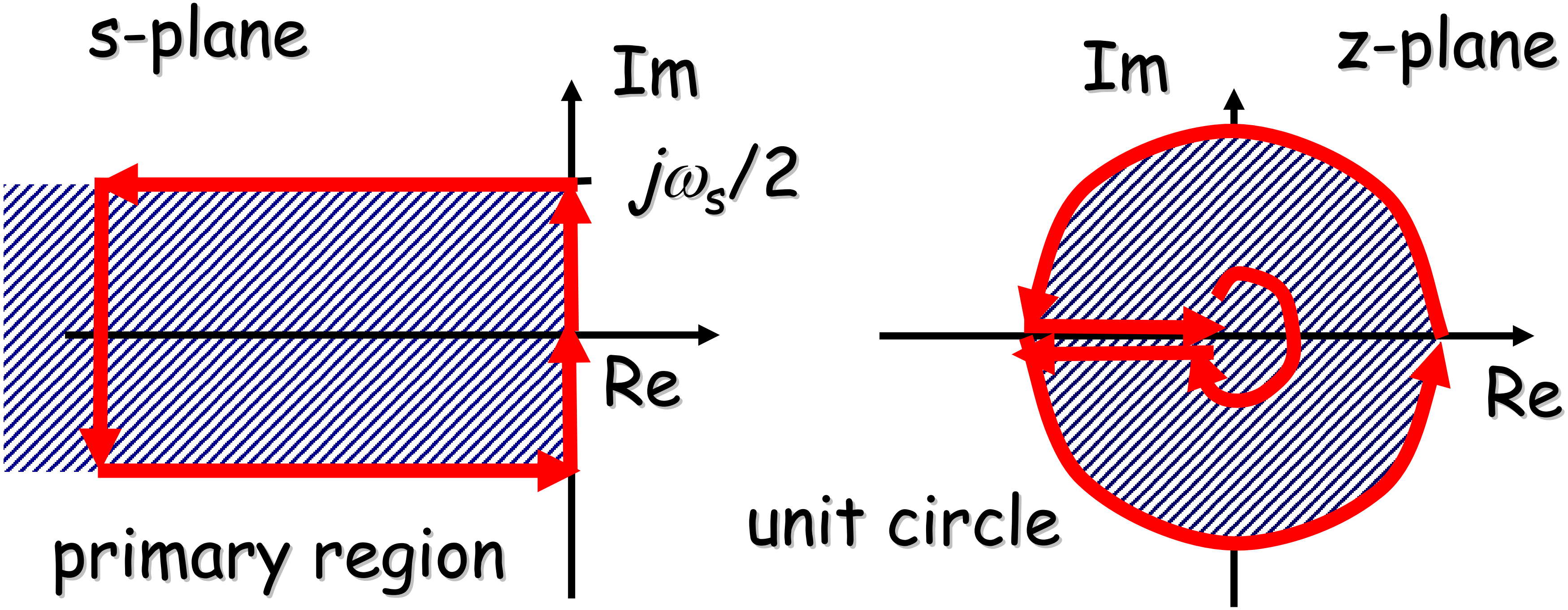
- Another way of obtaining these equations is by means of the z-transform

z-transform

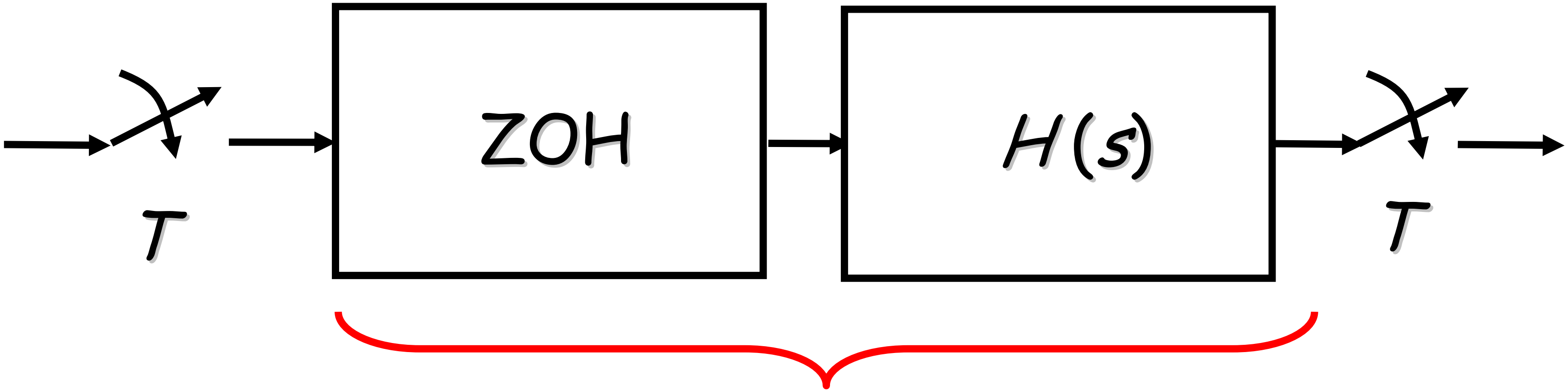
$$z = e^{sT} \quad \text{or} \quad z^{-1} = e^{-sT} \rightarrow \quad s = \frac{1}{T}\ln(z)$$

$$z^{-1} = e^{-j\omega T}$$

Compare dead time



z-transform



ZOH: $\frac{1-e^{-sT}}{s}$

$$\mathcal{Z}\left\{\frac{1-e^{-sT}}{s}H(s)\right\}$$

$$\mathcal{Z}\left\{\frac{H(s)}{s}\right\}\left(1-z^{-1}\right)=\frac{z-1}{z}\mathcal{Z}\left\{\frac{H(s)}{s}\right\}$$

z-transform

- In general z-transforms via
  - Table with transformations
  - Matlab: c2d
    - SYS_DISCR = cw2d(SYS_CONT,T,'zoh')
  - Approximations:
  - Euler, Tustin

Approximations Euler

s is the 'derivitative operator'

Discrete Euler approximation (forward form)

$$\frac{dx}{dt} \approx \frac{x\left[(k+1)T\right]-x(kT)}{T}$$

$$\boxed{s \to \frac{z-1}{T}} \quad \boxed{\frac{1}{s} \to \frac{T}{z-1}}$$

$$sX(s) = Y(s) \to \frac{x\left[(k+1)T\right]-x(kT)}{T} = y(kT)$$

$$\frac{(z-1)}{T}X(z) = Y(z) \to \frac{Y(z)}{X(z)} = \frac{z-1}{T}$$

Approximations Euler

Control Engineering
University of Twente

s is the 'derivitative operator'

Discrete Euler approximation (backward form)

$$\frac{dx}{dt} \approx \frac{x(kT) - x\left[(k-1)T\right]}{T}$$

$$\boxed{s \rightarrow \frac{z-1}{zT}} \quad \boxed{\frac{1}{s} \rightarrow \frac{zT}{z-1}}$$

$$sX(s) = Y(s) \rightarrow \frac{x(kT) - x\left[(k-1)T\right]}{T} = y(kT)$$

$$\frac{\left(1 - z^{-1}\right)}{T} X(z) = Y(z) \rightarrow \frac{Y(z)}{X(z)} = \frac{z-1}{zT} = \frac{1}{z}\frac{z-1}{T}$$

Approximations Tustin

Forward Euler approximation

$$\frac{x\left[(k+1)T\right]-x(kT)}{T}=y(kT)$$

Backward Euler approximation

$$\frac{x\left[(k+1)T\right]-x(kT)}{T}=y\left[(k+1)T\right]$$

Mean value:

$$\frac{x\left[(k+1)T\right]-x(kT)}{T}=\frac{y\left[(k+1)T\right]+y(kT)}{2}$$

Approximations Tustin

Mean value:

$$\frac{x\left[(k+1)T\right]-x(kT)}{T}=\frac{y\left[(k+1)T\right]+y(kT)}{2}$$

$$\frac{z-1}{T}X(z)=\frac{z+1}{2}Y(z)$$

$$\frac{Y(z)}{X(z)}=\frac{2}{T}\frac{z-1}{z+1}$$

$$\boxed{s\to\frac{2}{T}\frac{z-1}{z+1}}$$

Aproximations

$$\boxed{\text{Forward Euler}\quad s \rightarrow \frac{z-1}{T}}$$

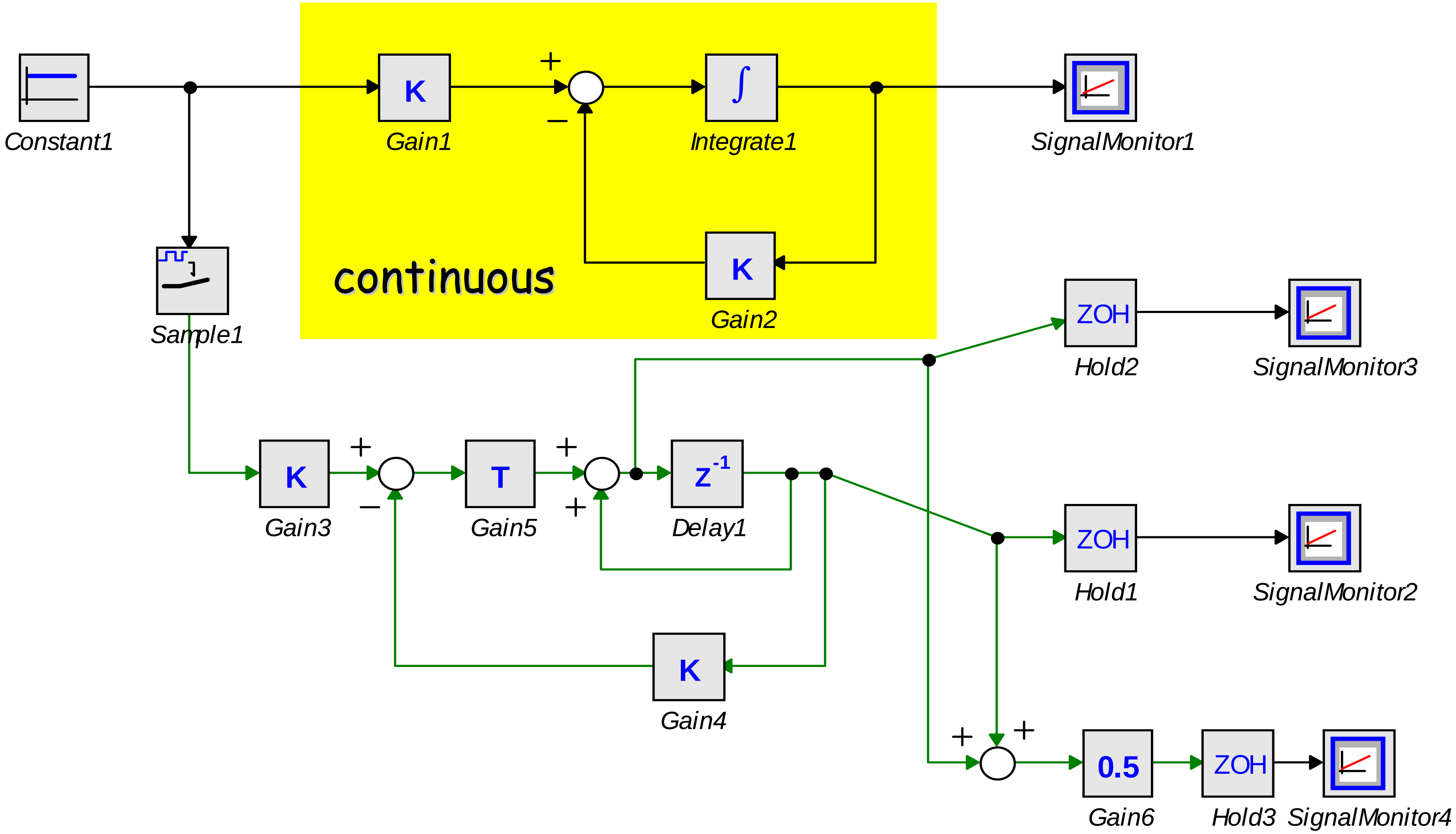
$$\boxed{\frac{1}{s} \rightarrow \frac{T}{z-1}}$$

$$\boxed{\text{Backward Euler}\quad s \rightarrow \frac{1}{z}\frac{z-1}{T}}$$

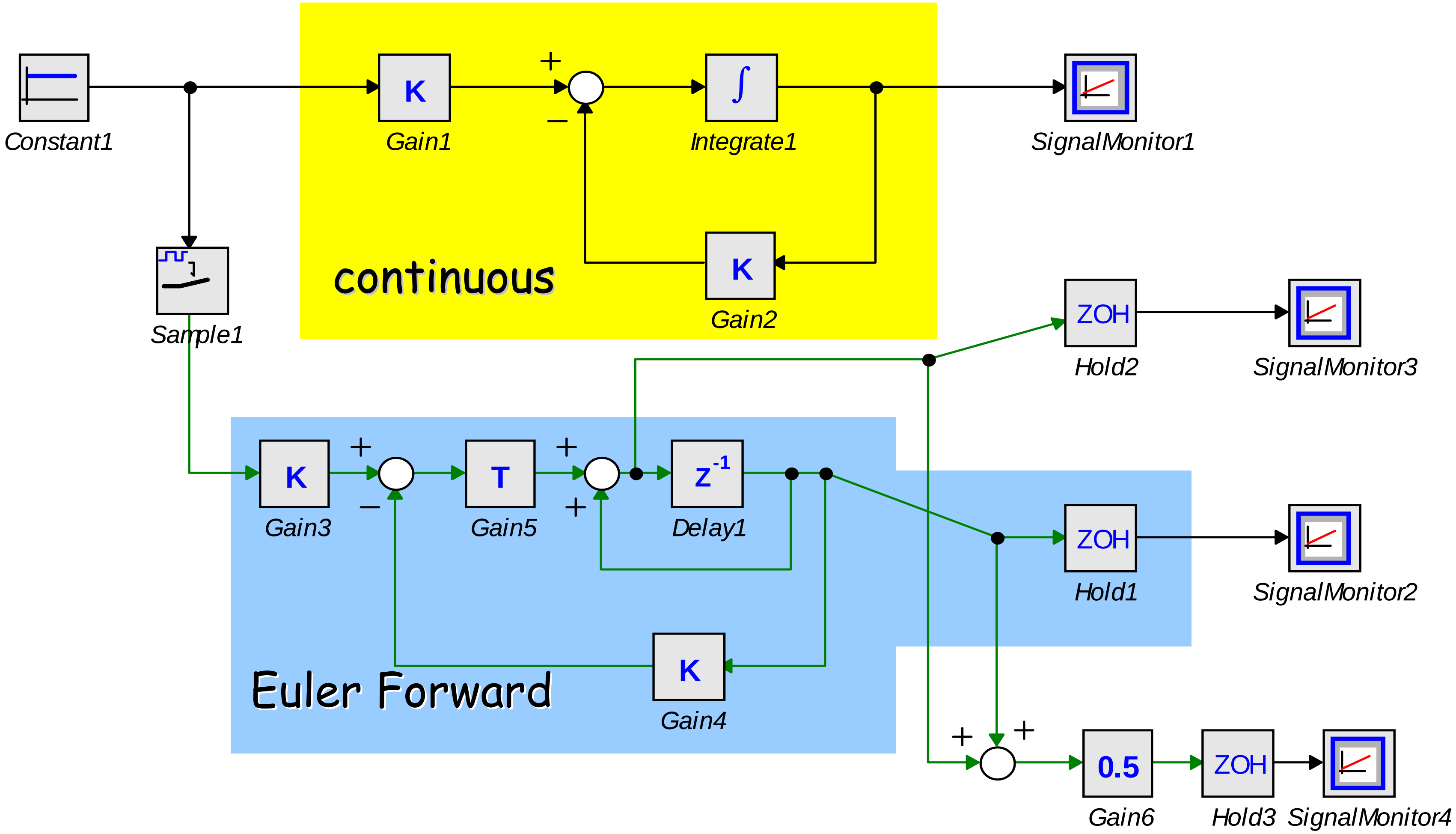
$$\boxed{\frac{1}{s} \rightarrow \frac{zT}{z-1}}$$

$$\boxed{\text{Tustin (bilinair)}\quad s \rightarrow \frac{2}{T}\frac{z-1}{z+1}}$$

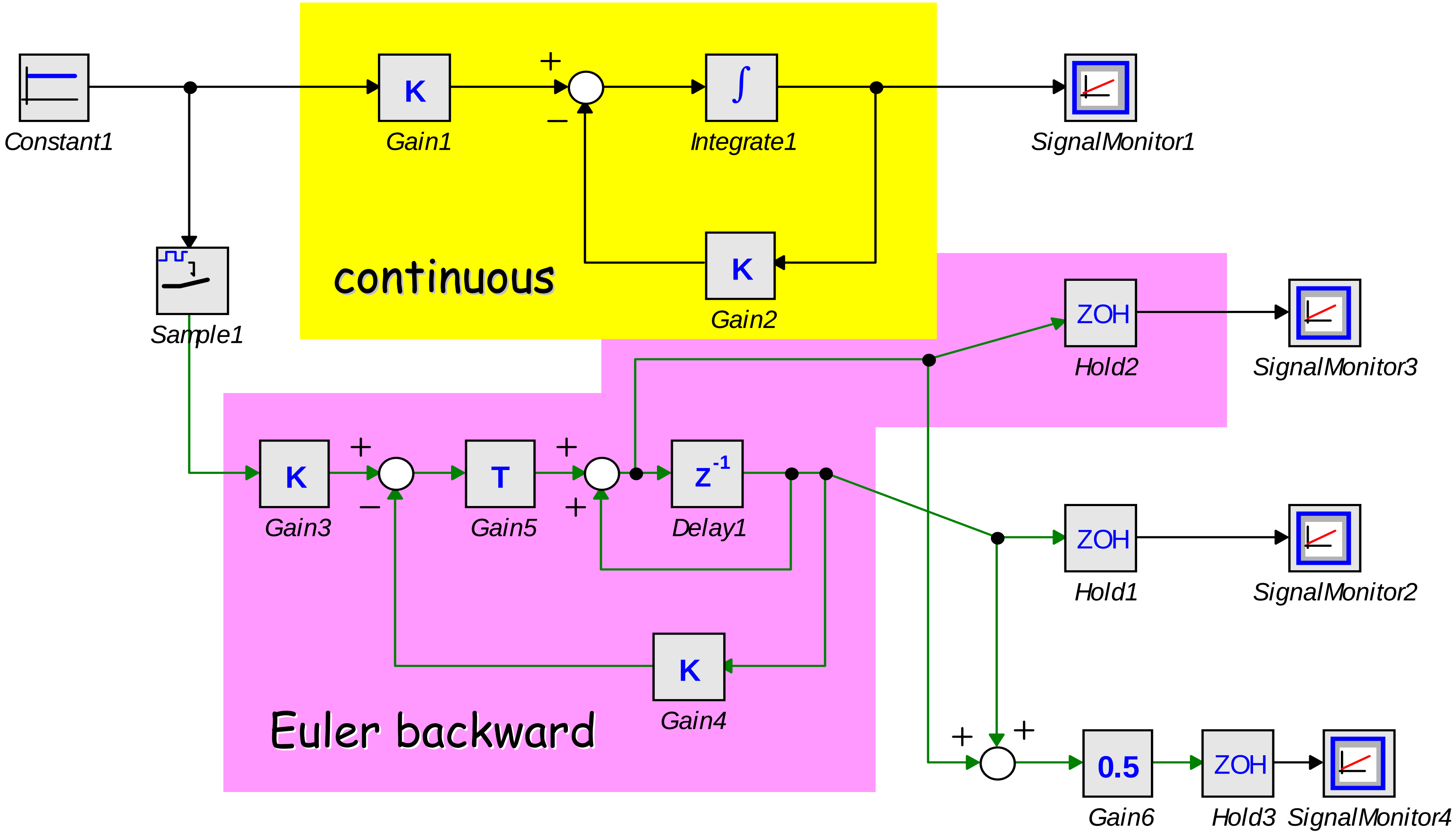
Example


Constant1
K
Gain1
+
–
∫
Integrate1
SignalMonitor1
K
Gain2
continuous
Sample1
ZOH
Hold2
SignalMonitor3
K
Gain3
+
–
T
Gain5
+
+
z^{-1}
Delay1
ZOH
Hold1
SignalMonitor2
K
Gain4
+
+
0.5
Gain6
ZOH
Hold3
SignalMonitor4

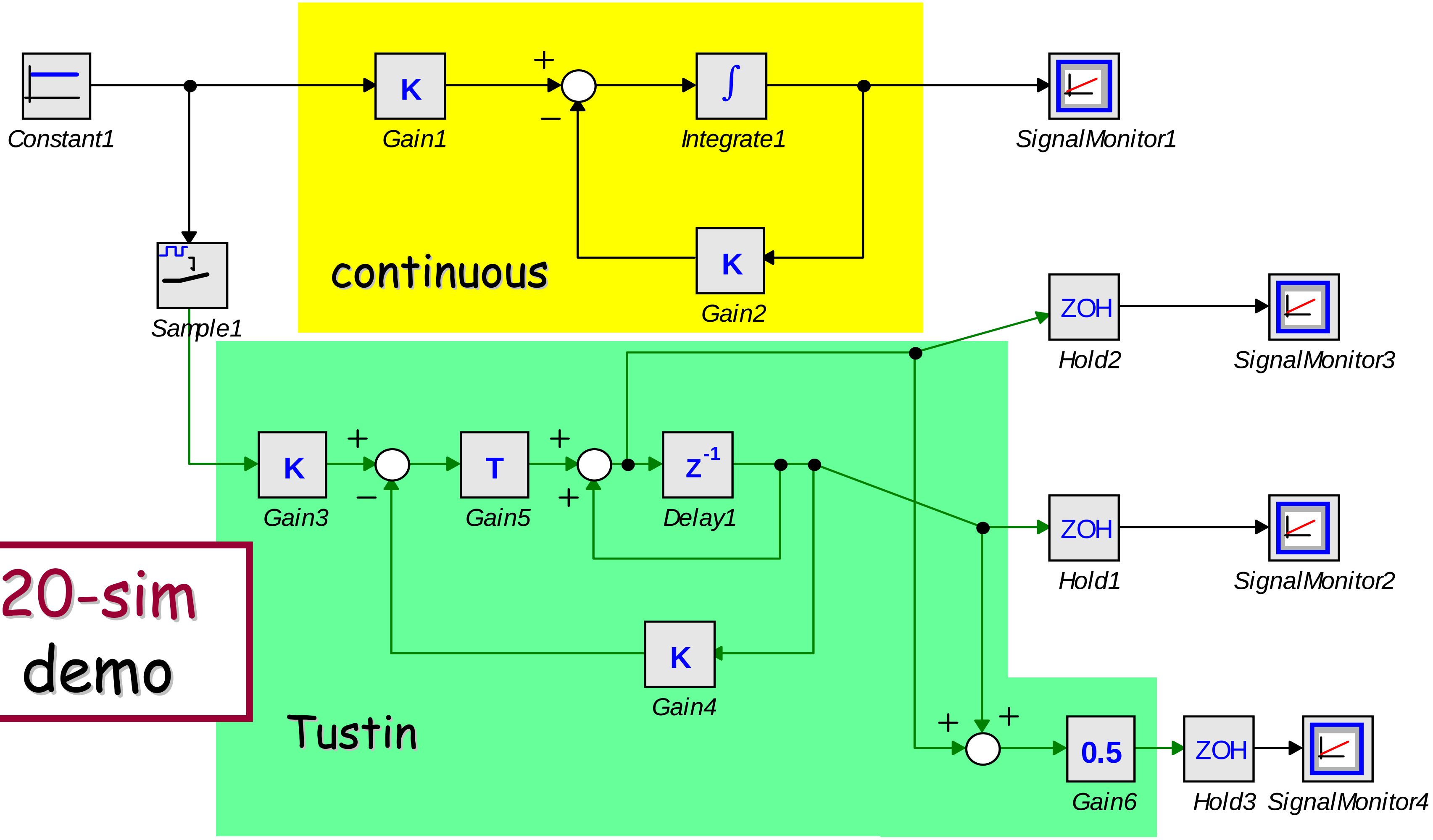
Example


Constant1
K
Gain1
+
−
∫
Integrate1
SignalMonitor1
K
Gain2
continuous
Sample1
ZOH
Hold2
SignalMonitor3
K
Gain3
+
−
T
Gain5
+
+
z-1
Delay1
ZOH
Hold1
SignalMonitor2
K
Gain4
Euler Forward
+
+
0.5
Gain6
ZOH
Hold3
SignalMonitor4

Example


Constant1
K
Gain1
+
−
∫
Integrate1
SignalMonitor1
Sample1
continuous
K
Gain2
ZOH
Hold2
SignalMonitor3
K
Gain3
+
−
T
Gain5
+
+
z^{-1}
Delay1
ZOH
Hold1
SignalMonitor2
K
Gain4
Euler backward
+
+
0.5
Gain6
ZOH
Hold3
SignalMonitor4

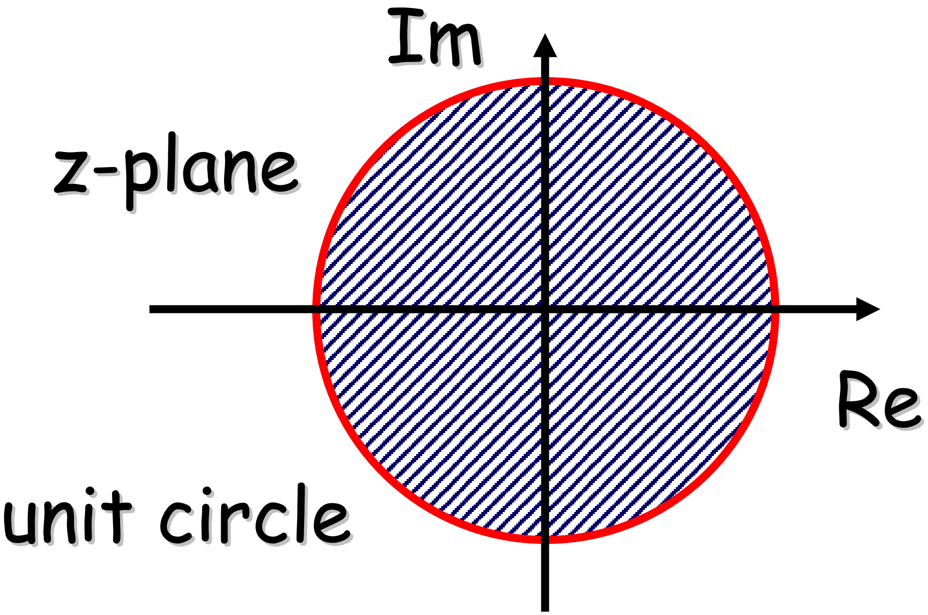
Example



20-sim
demo

Stability

- A digital control system is stable when the poles are within the unit circle



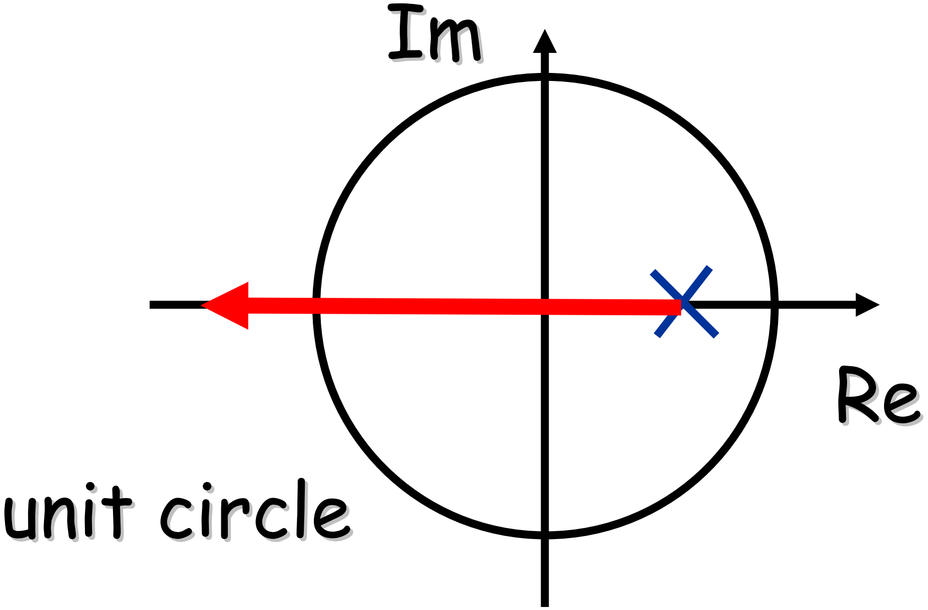
Design in the z-plane ?

- z-plane not so easy for the design
- All root locus rules remain valid

First-order system

First-order system

$$H_L = \frac{X(z)}{U(z)} = \frac{K[1-e^{-aT}]}{z-e^{-aT}}$$



Closed loop can become unstable !!

First-order system

First-order system $$\frac{X(z)}{U(z)} = \frac{K[1-e^{-aT}]}{z-e^{-aT}}$$

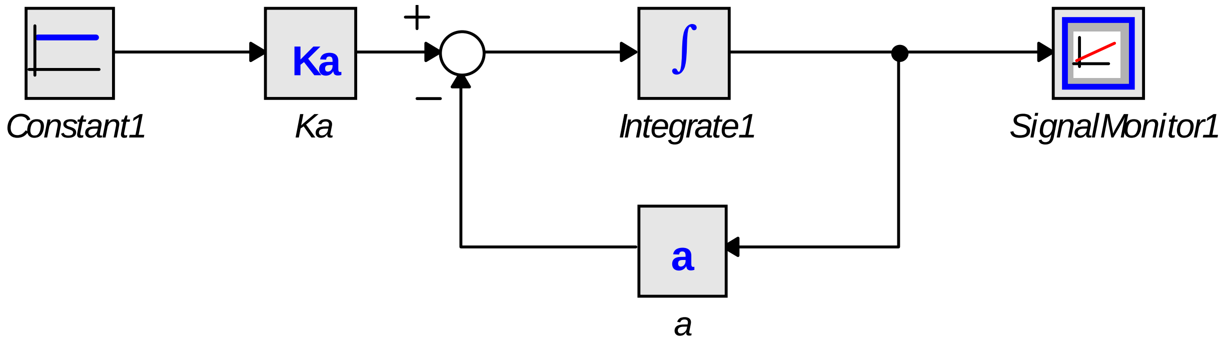
Making T small improves the stability:

$K[1-e^{-aT}]$ becomes smaller

But for T too small:

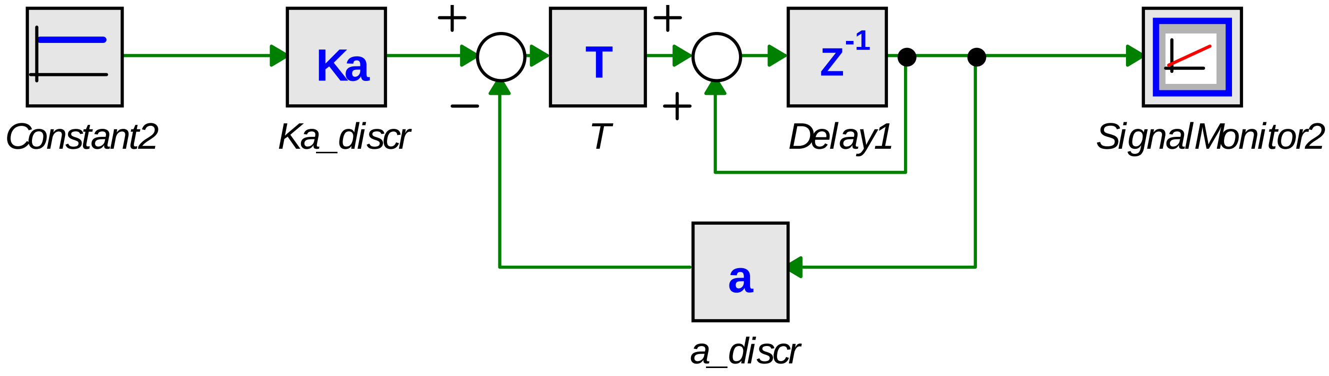
Numerical problems

Z-transform versus Euler



$$\frac{dx}{dt} = -ax + Kau$$

$$x(k+1) = (1 - aT)x(k) + KaTu(k)$$



Z-transform versus Euler

$$\text{ZOH: } \frac{X(z)}{U(z)} = \frac{K(1-e^{-aT})}{z-e^{-aT}}$$

$$x(k+1) = e^{-aT} + K(1-e^{-aT})$$

1e orde Taylor:

$$x(k+1) = (1-aT)x(k) + K(1-(1-aT))u(k)$$

$$= (1-aT)x(k) + KaTu(k)$$

$$\text{Euler: } x(k+1) = (1-aT)x(k) + KaTu(k)$$

Hard Real Time

We notice that

- the sampling time T is a parameter in discrete time systems:
- T must be constant
- Hard real time

$$\text{ZOH: } \frac{X(z)}{U(z)} = \frac{K(1-e^{-aT})}{z-e^{-aT}}$$

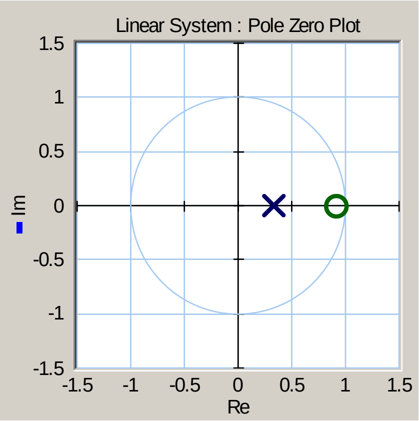
Comparison

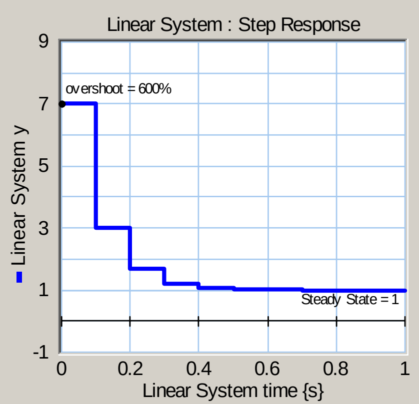
$$10\frac{s+1}{s+10}$$

Tustin - Euler – ‘ZOH’ (T=0.1)

Tustin

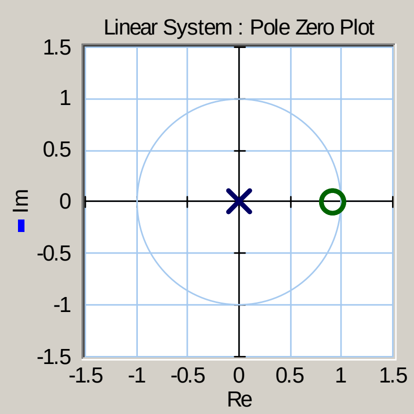
$$0.7\frac{10z-9.048}{z-0.333}$$

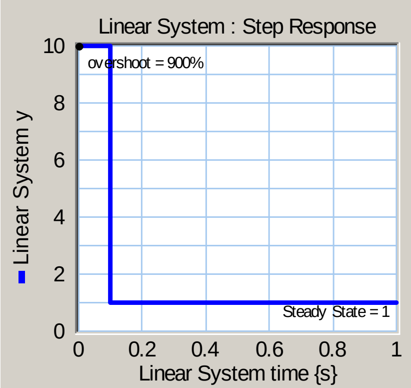




Euler

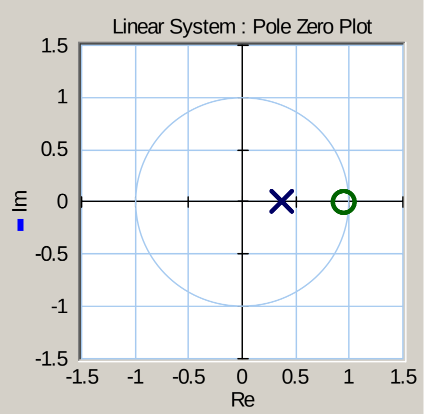
$$\frac{10z-9}{z}$$

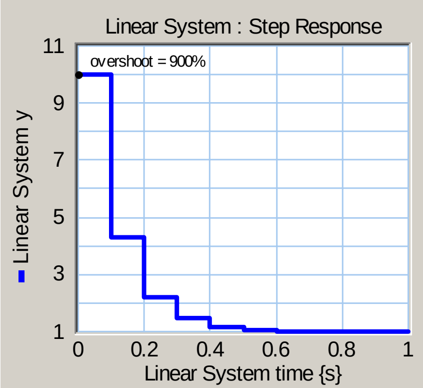




‘ZOH’

$$\frac{10z-9.368}{z-0.3679}$$

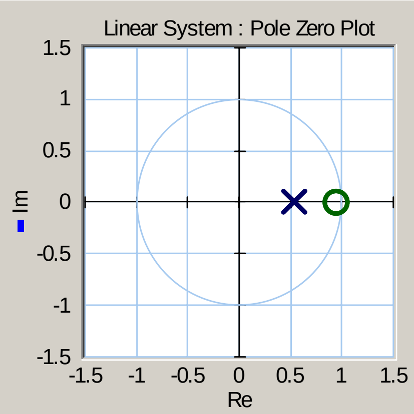


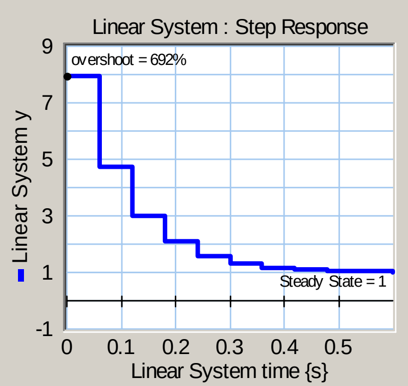


Tustin - Euler - ‘ZOH’ (T=0.06)

Tustin

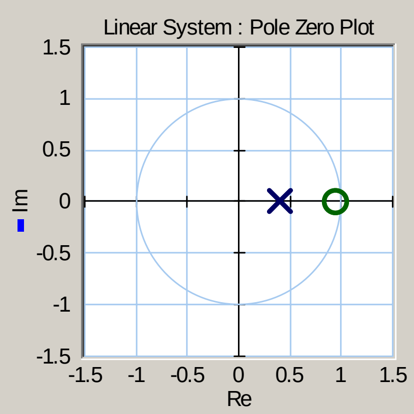
$$0.7923\frac{10z-9.417}{z-0.5385}$$

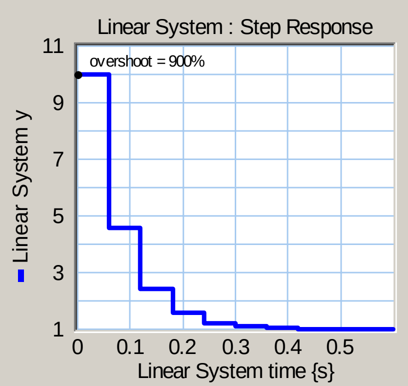




Euler

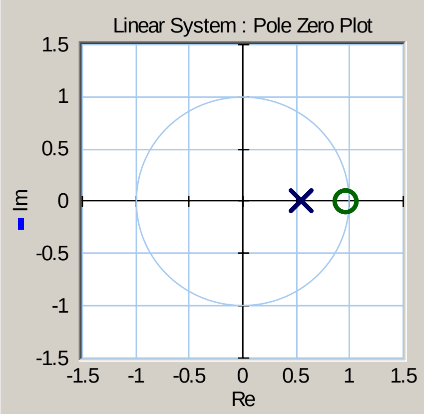
$$\frac{10z-9.4}{z-0.4}$$

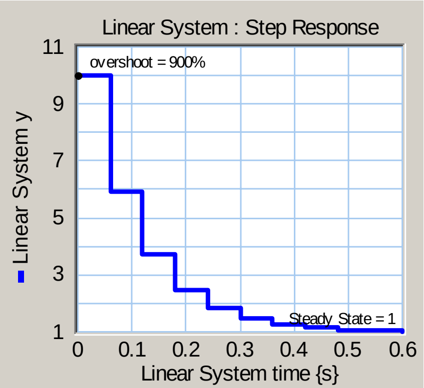




‘ZOH’

$$\frac{10z-9.549}{z-0.5488}$$

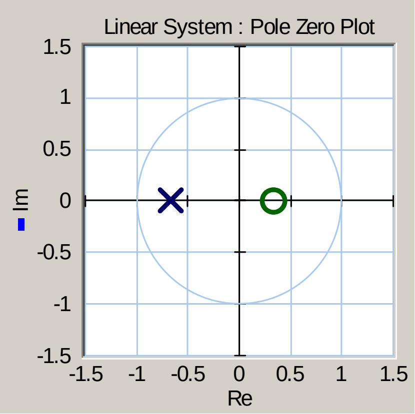


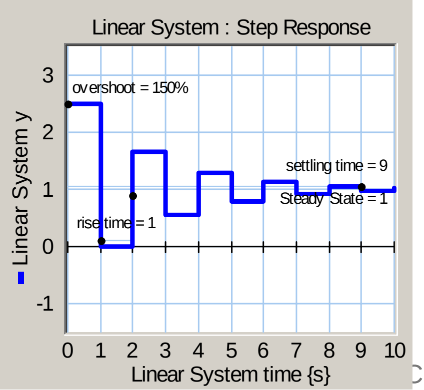


Tustin - Euler - ‘ZOH’ (T=1)

Tustin

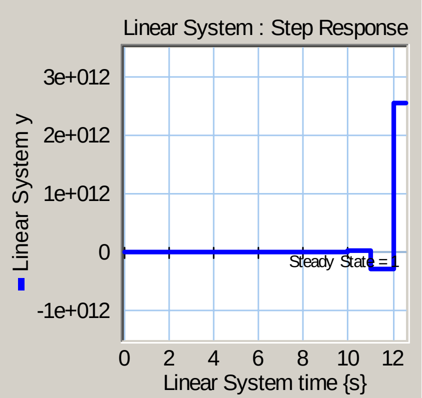
$$0.25\frac{10z-3.333}{z-0.6676}$$

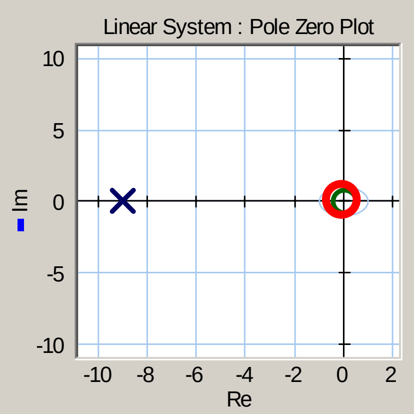




Euler

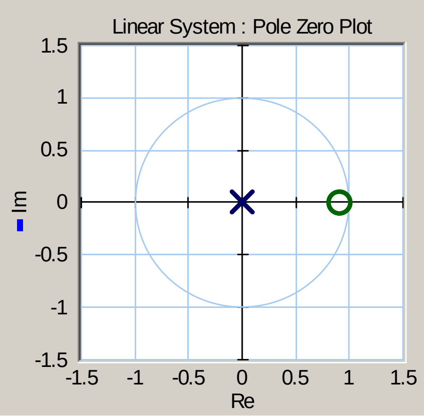
$$\frac{10z}{z+9}$$

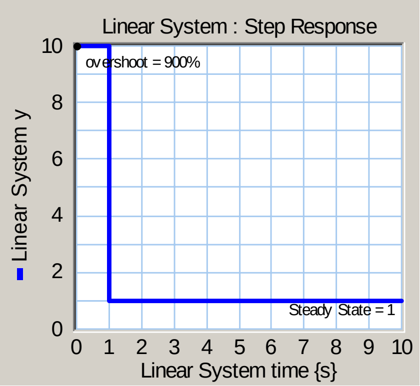




‘ZOH’

$$\frac{10z-9}{z}$$

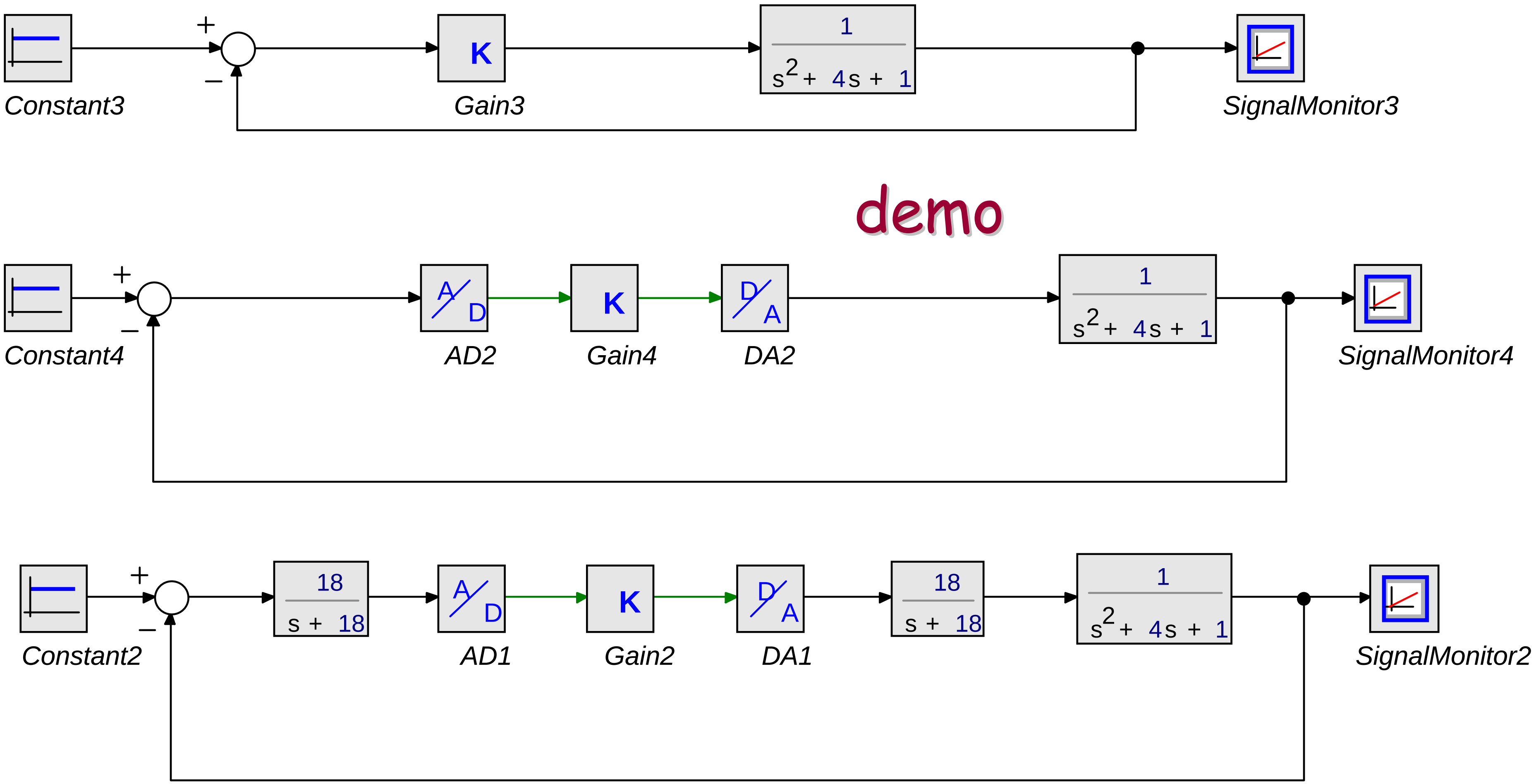




Rules of thumb

- Select $\omega_s \approx 10\ \omega_{n,max}$
  - where $\omega_{n,max}$ is location of smallest time constant in the (total) system
- Use an anti alias filter with cut-off frequency ω_c
- $\omega_b \ll \omega_c < 0.5\omega_s \quad (\omega_b \ll \omega_c < 5\omega_b)$

Example



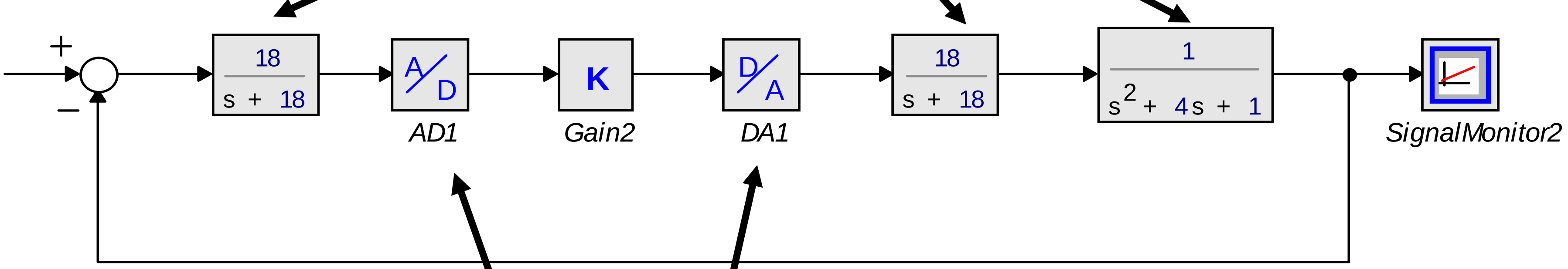
demo

Sampletime ≠ step size

Can be constant or variable

Stepsize 0.01s

Determined by numerical stability of simulation



Must be constant

Sampletime 0.14s

Determined by Nyquist criterium

Conclusions

- Z-transform describes continous-time systems as seen by the computer
- Sampling introduces amplitude disretisation and discrete time
- Sampling rate influences stability of computer controlled systems

Conclusions

- Sampling rate should not be chosen too small because of:
  - Stability
  - Signal reconstruction