

Non-linear systems

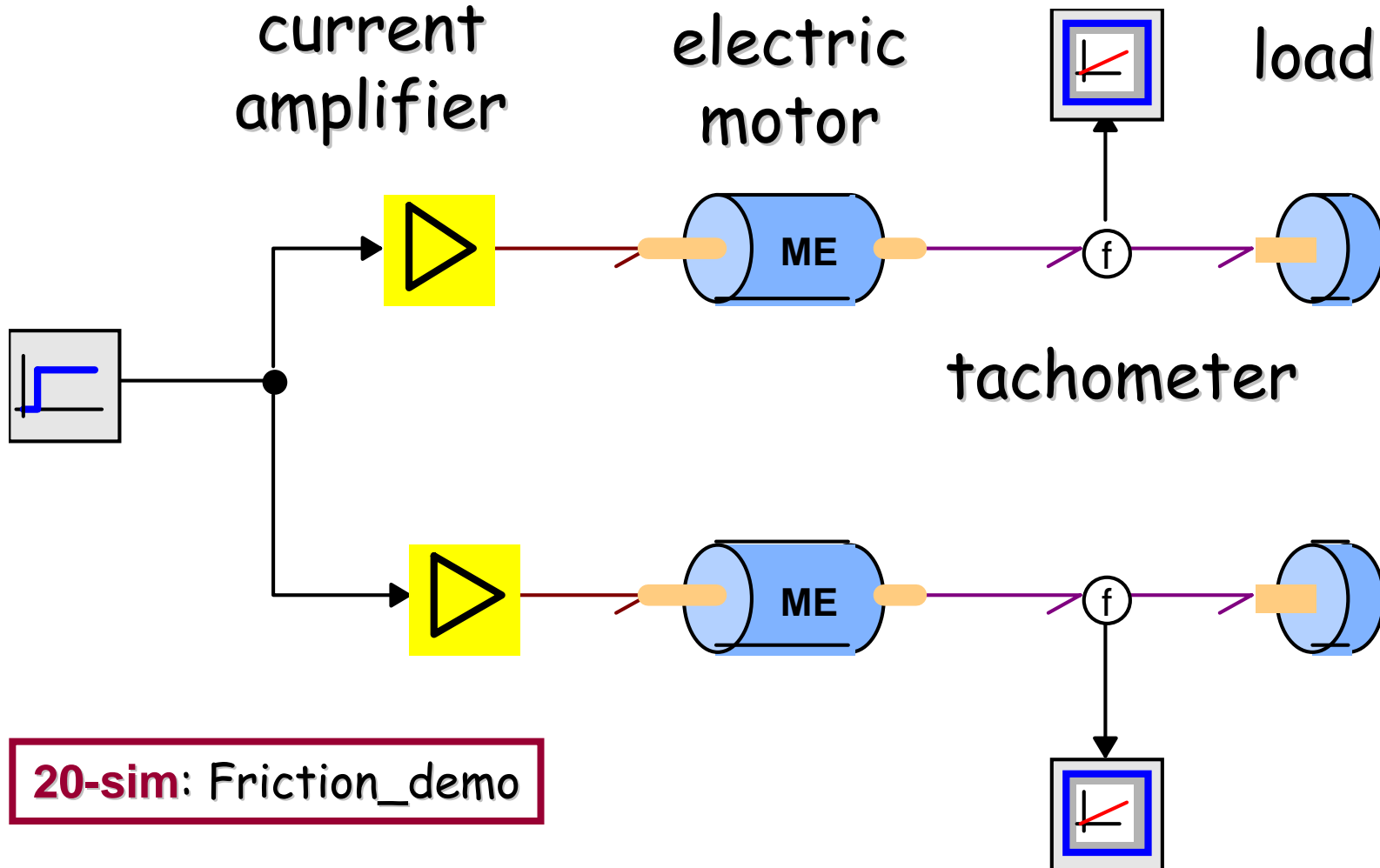
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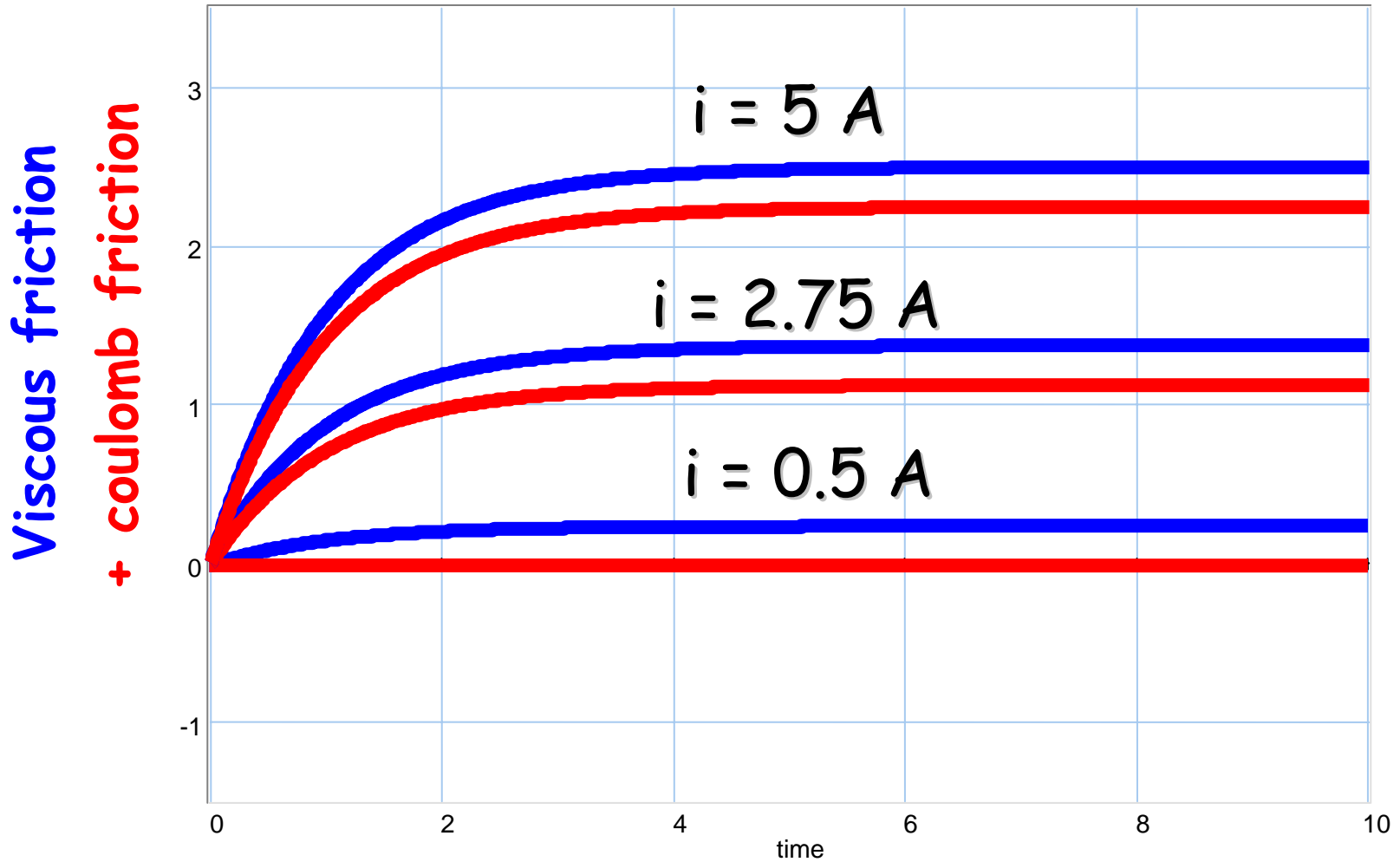
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- Various non-linearities
- reasons for the presence of non-linear elements
- analysis in the phase plane
- analysis with describing functions

- Friction in a mechanical system
- Saturation in an amplifier
- Switching elements, e.g. in thermostats
- operating-point-dependent parameters

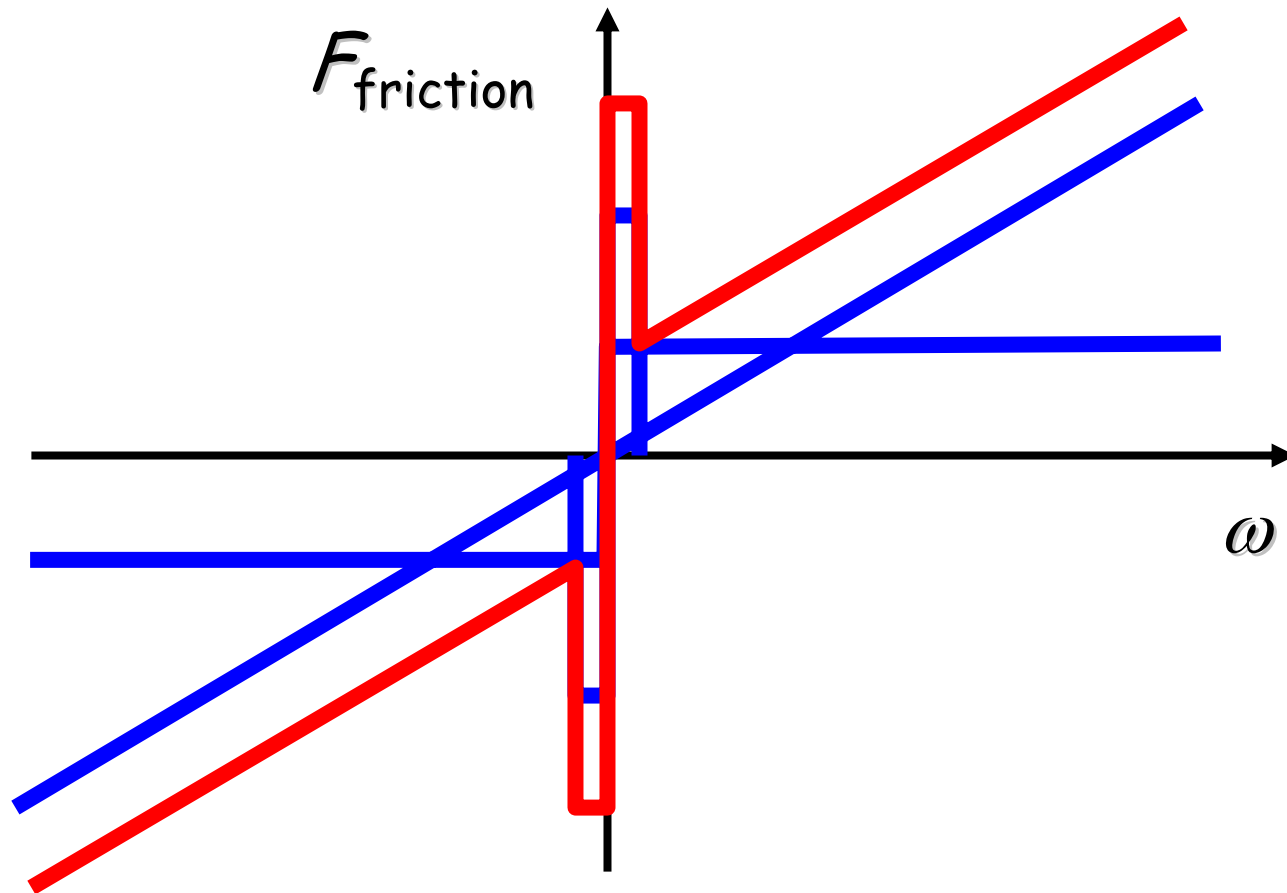


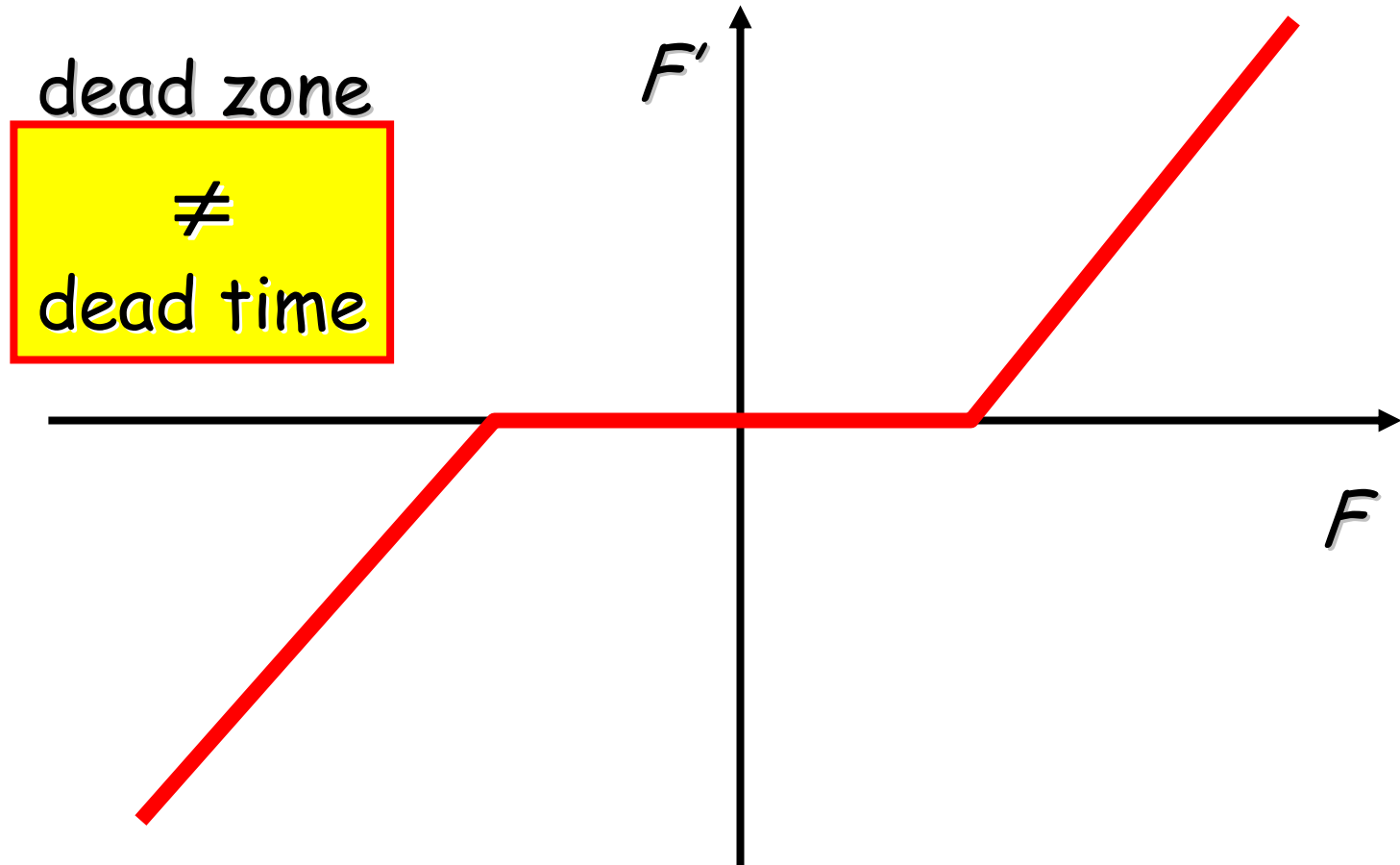
20-sim: Friction_demo



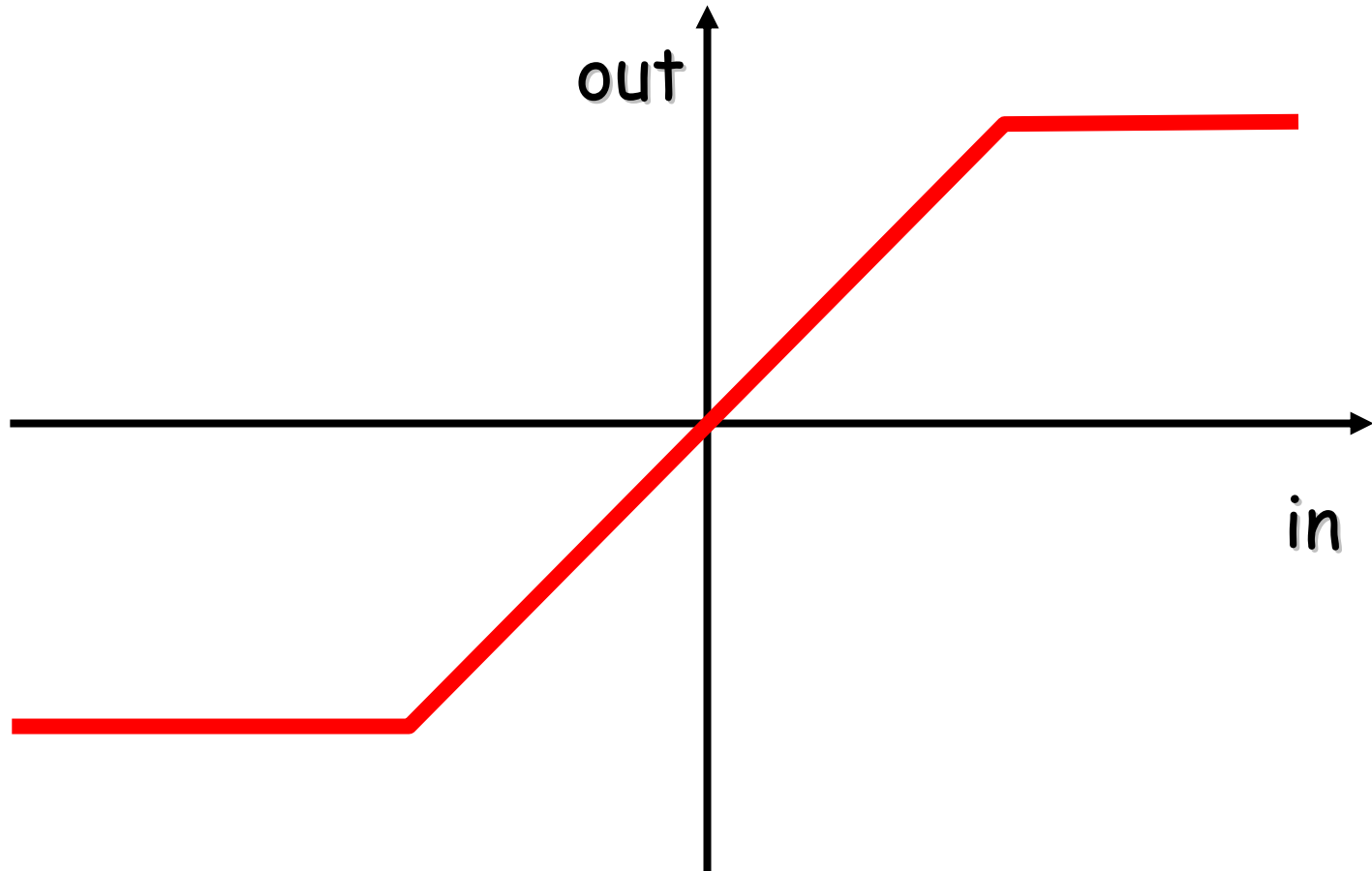
- For a non-linear system the output can be zero for small input changes
- If the input signal increases with a factor α , the change in the output can be smaller or larger than α .

Friction characteristic

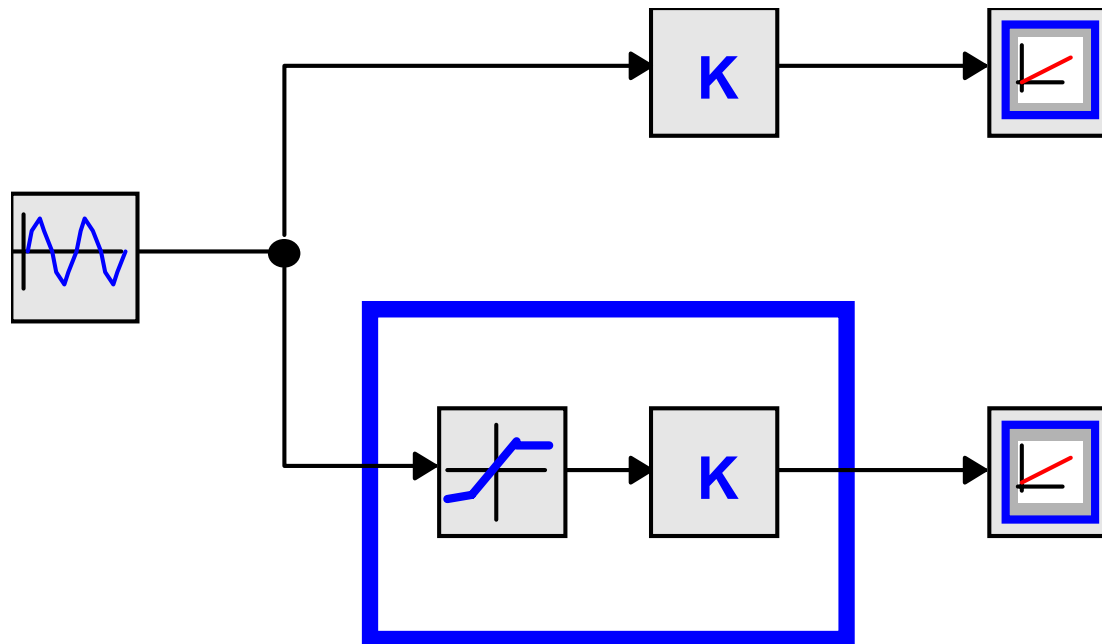




- Due to
 - maximum valve opening reached
 - output voltage of an amplifier limited by voltage of power supply
 - end stop

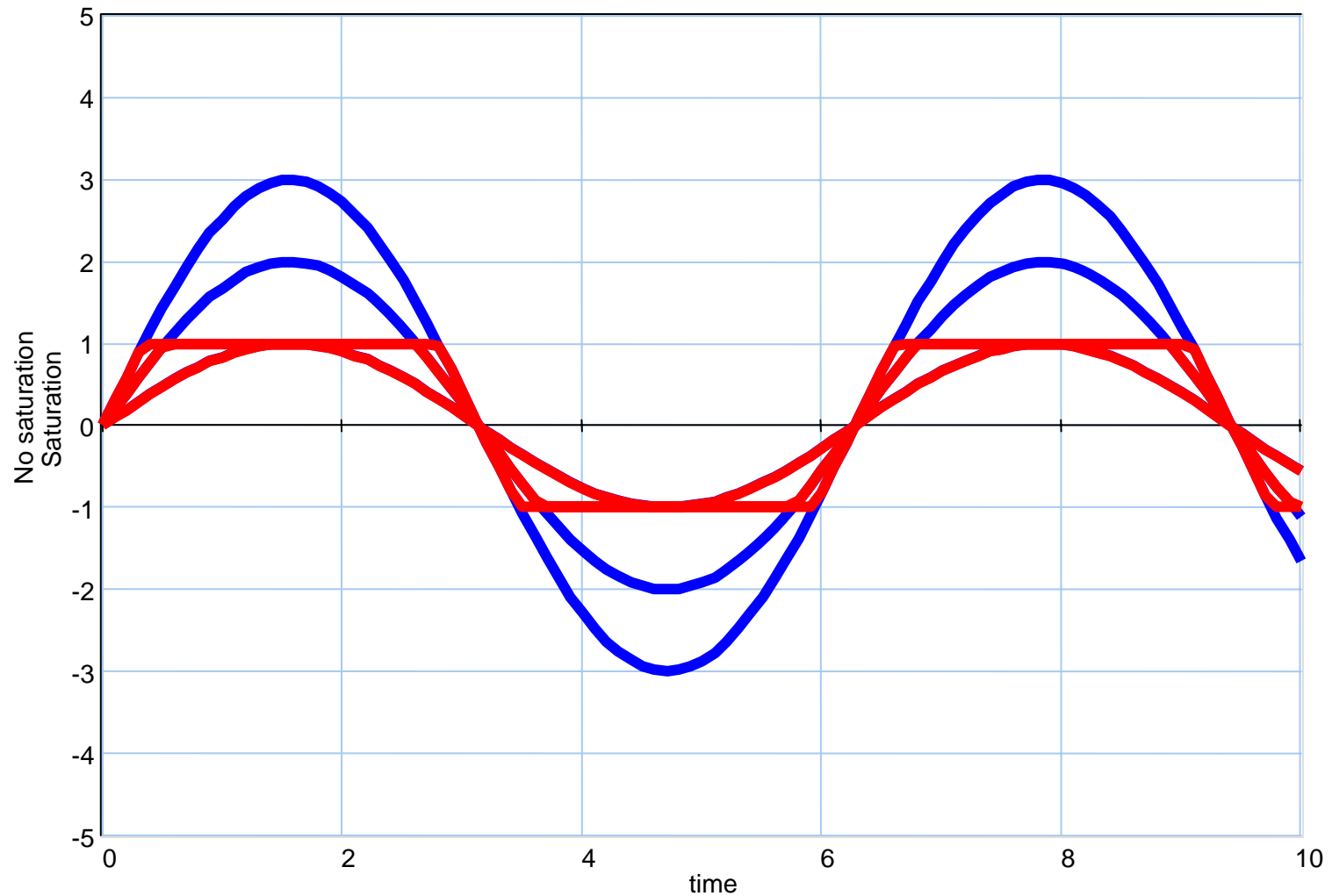


Saturation in amplifier

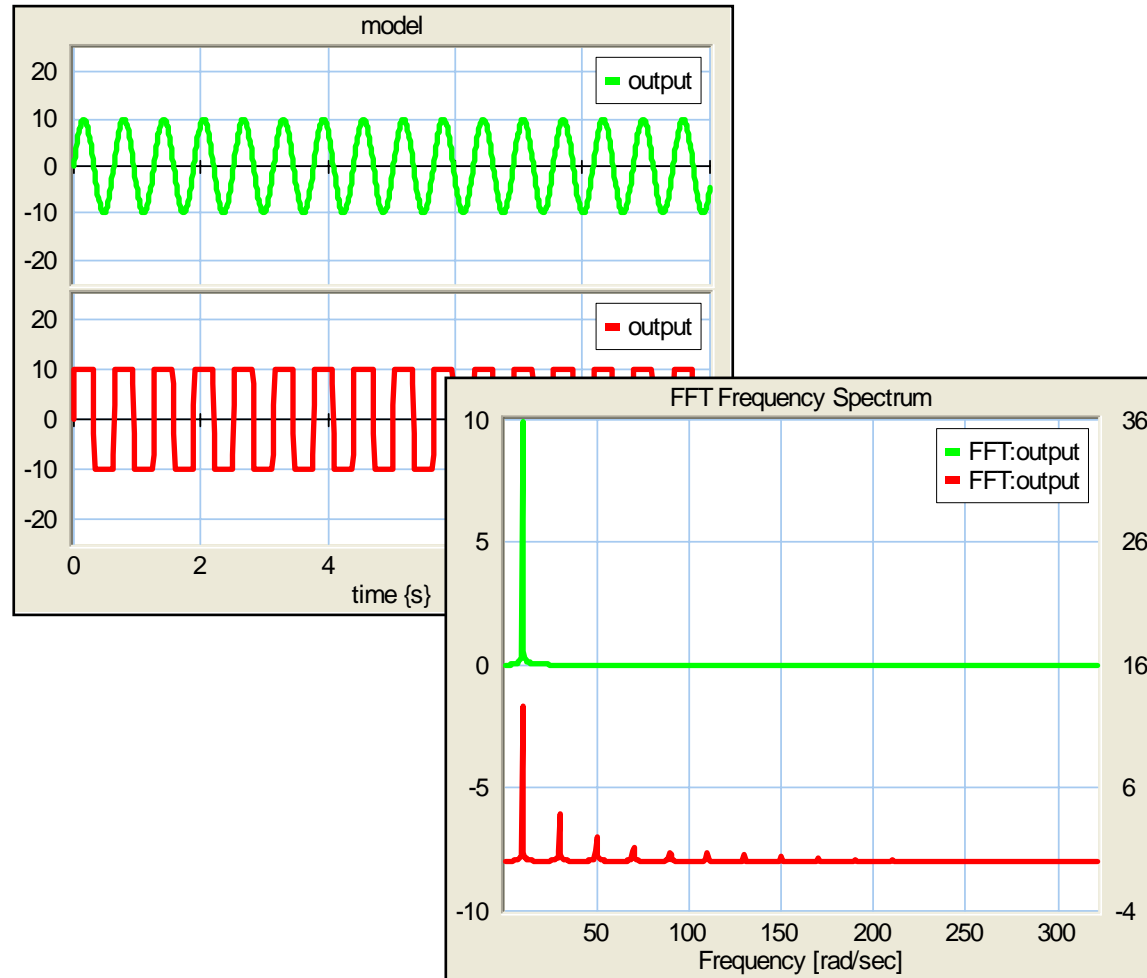


20-sim: Saturation_demo

Saturation in amplifier

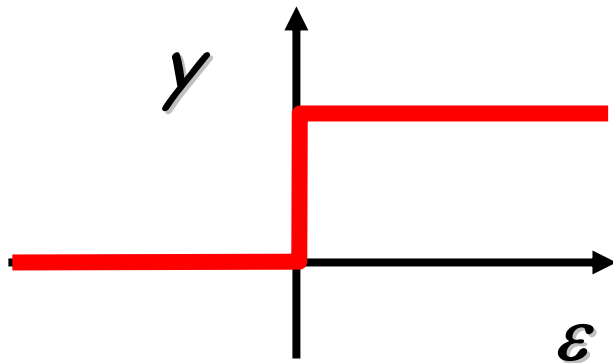


Saturation in amplifier

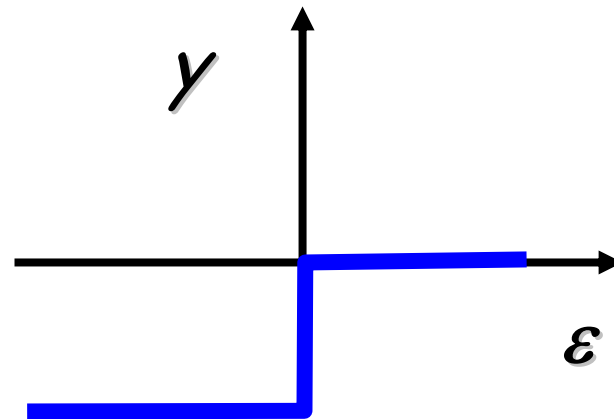


20-sim

- Temperature control in
 - buildings
 - boilers
 - refrigerators
 - cars (motor cooling)



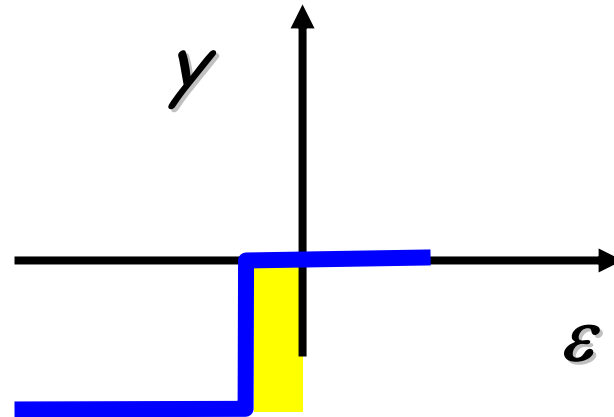
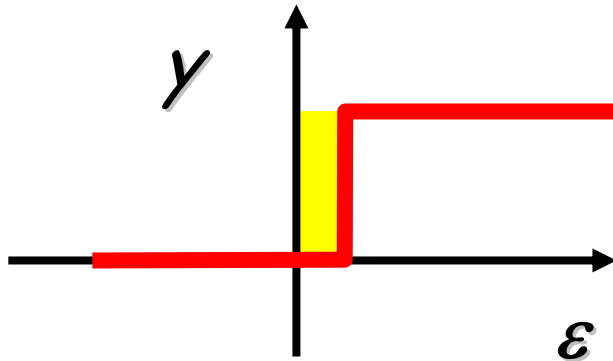
relay:
if $x > 0$, $y = y_{\max}$
else $y = 0$



different control systems
for heating and cooling

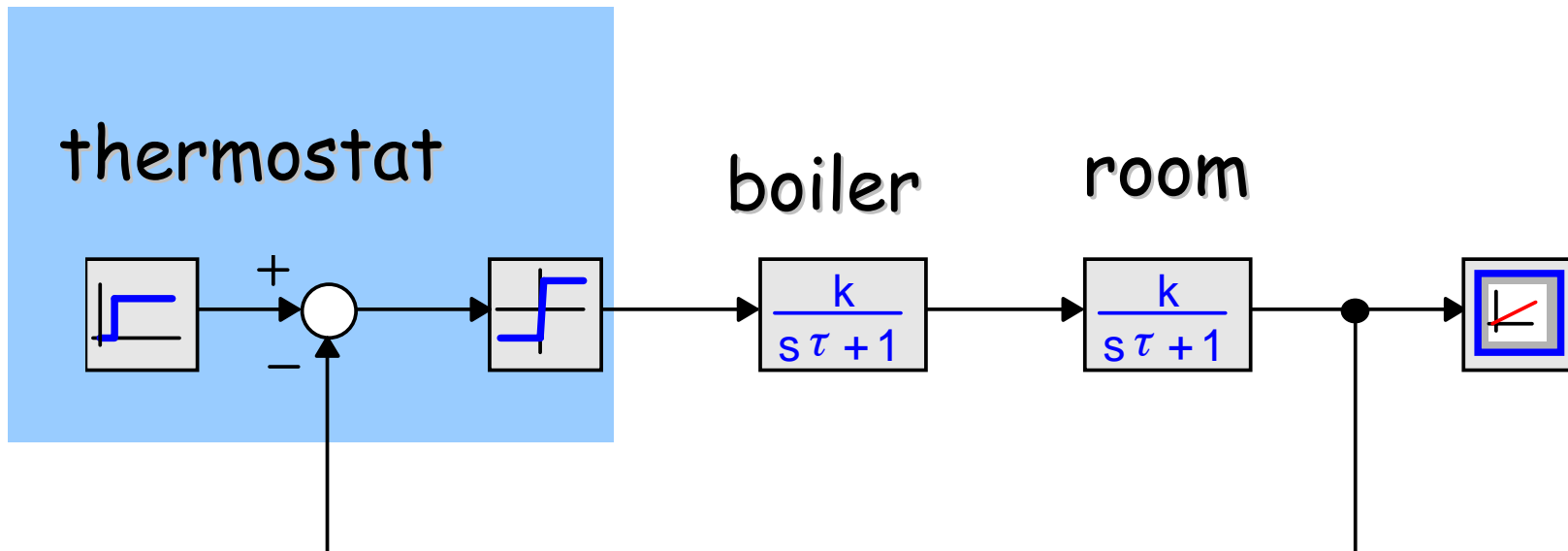
It makes no sense to damp the overshoot caused by the heating systems by switching on the air conditioning

In practice only one of the two systems active

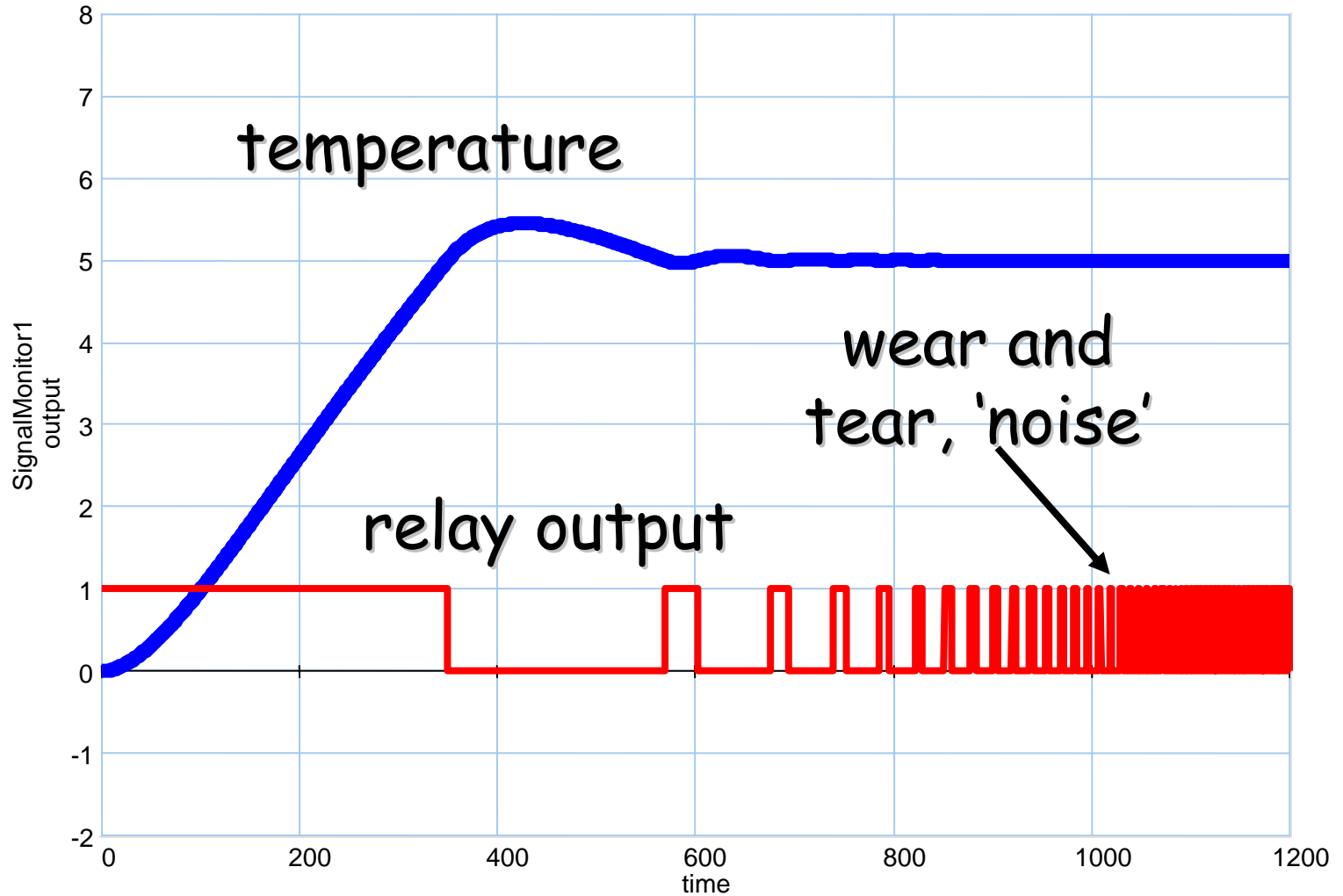


It makes no sense to damp the overshoot caused by the heating systems by switching on the air conditioning

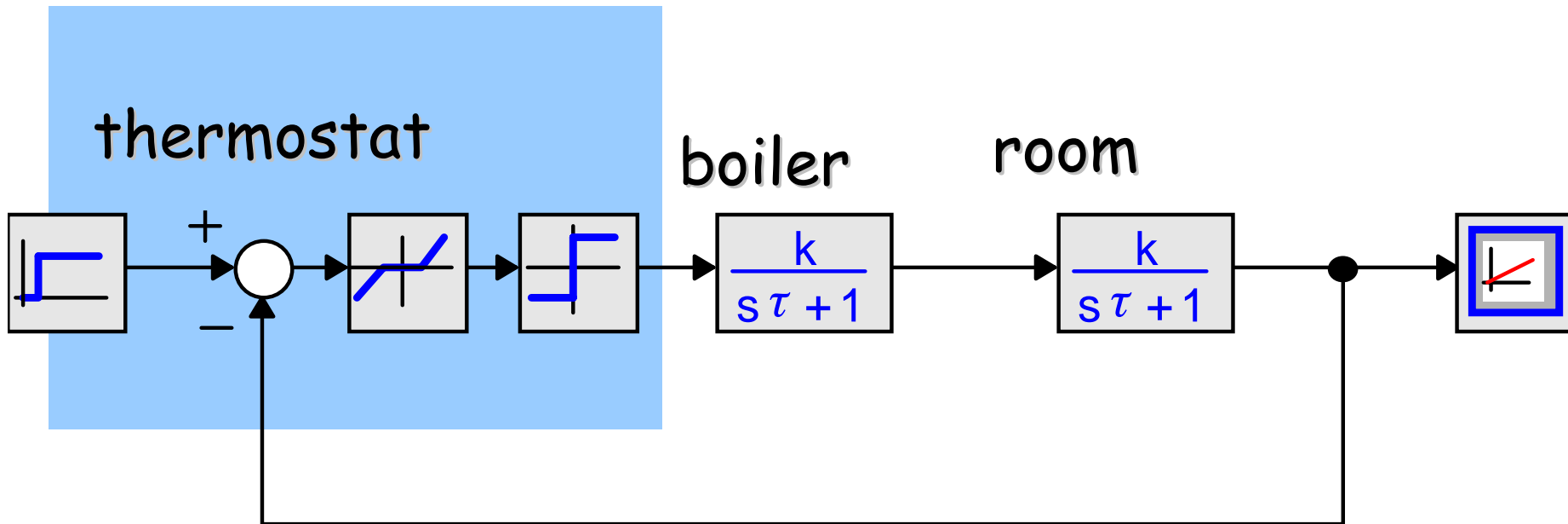
In practice only one of the two systems active



20-sim: Thermostat_demo1

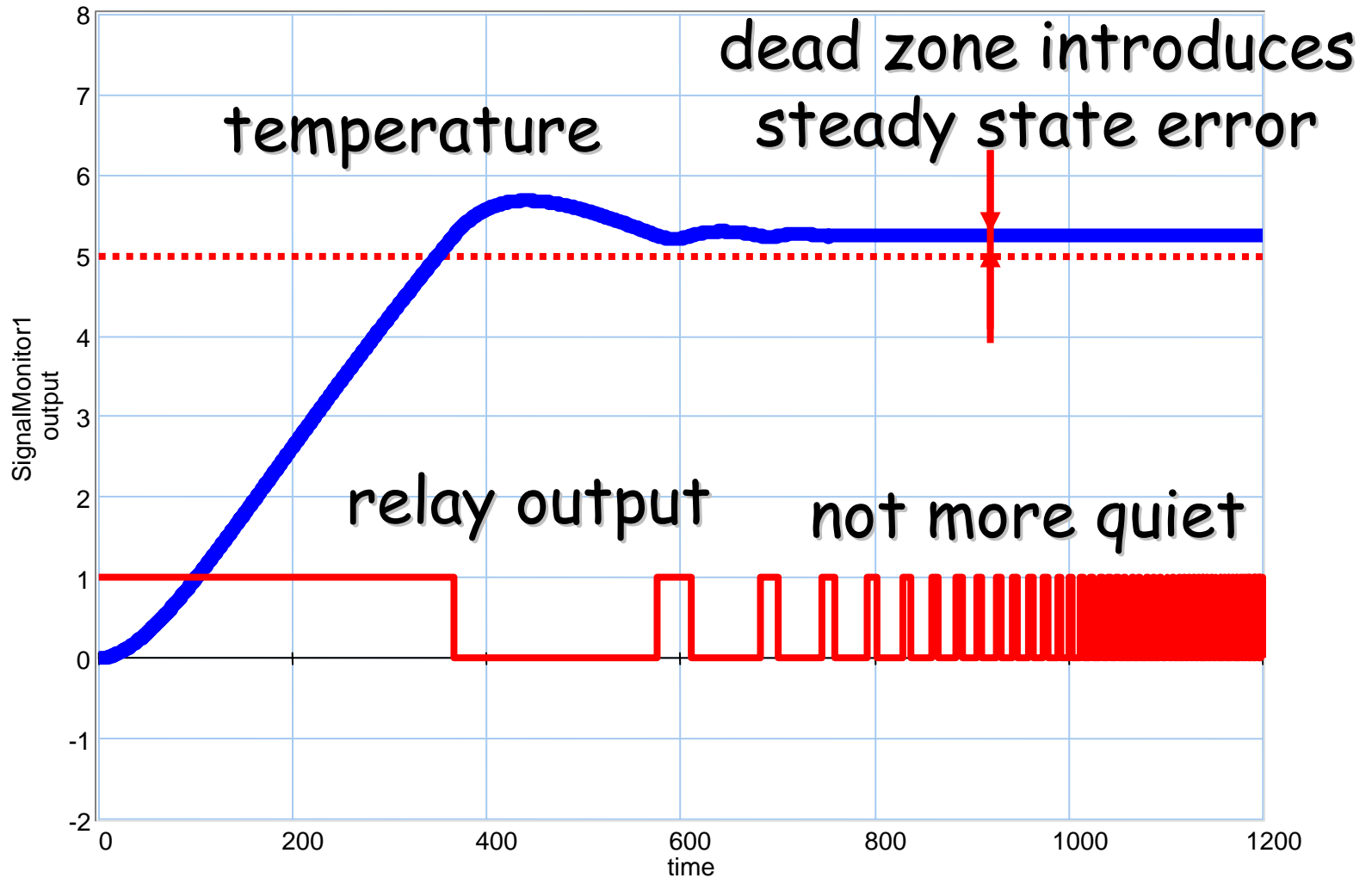


Relay plus dead zone

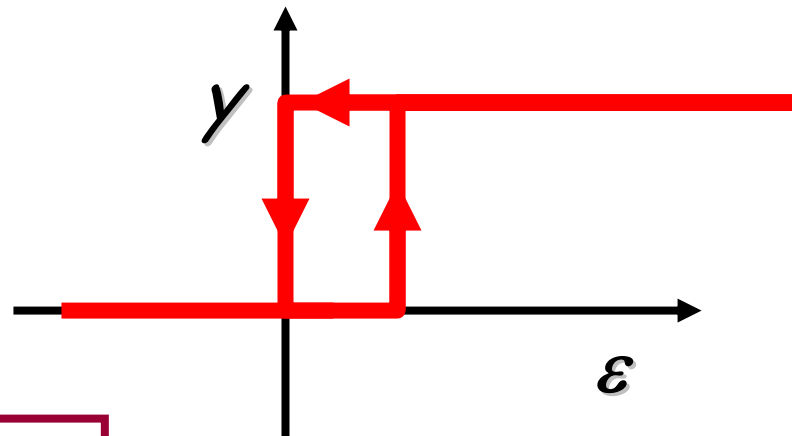
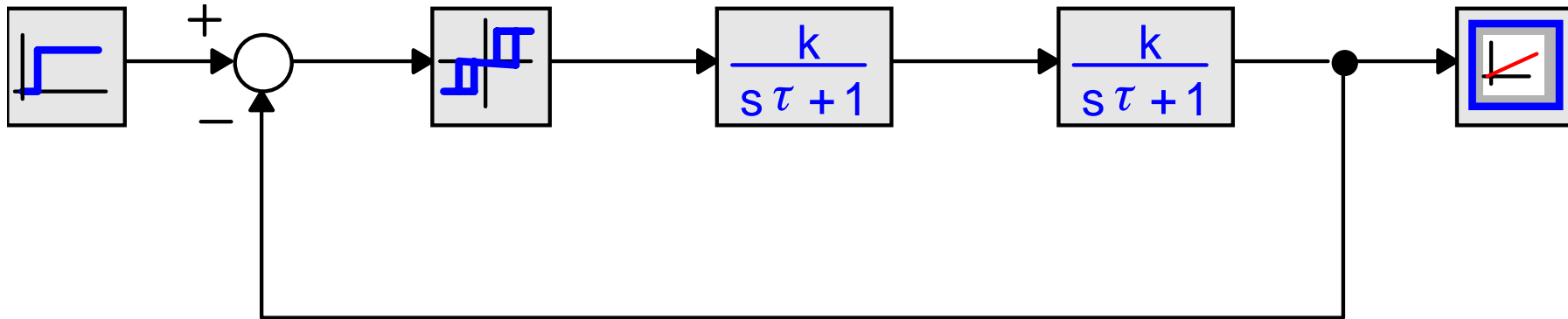


20-sim: Thermostat_demo2

Relay plus dead zone

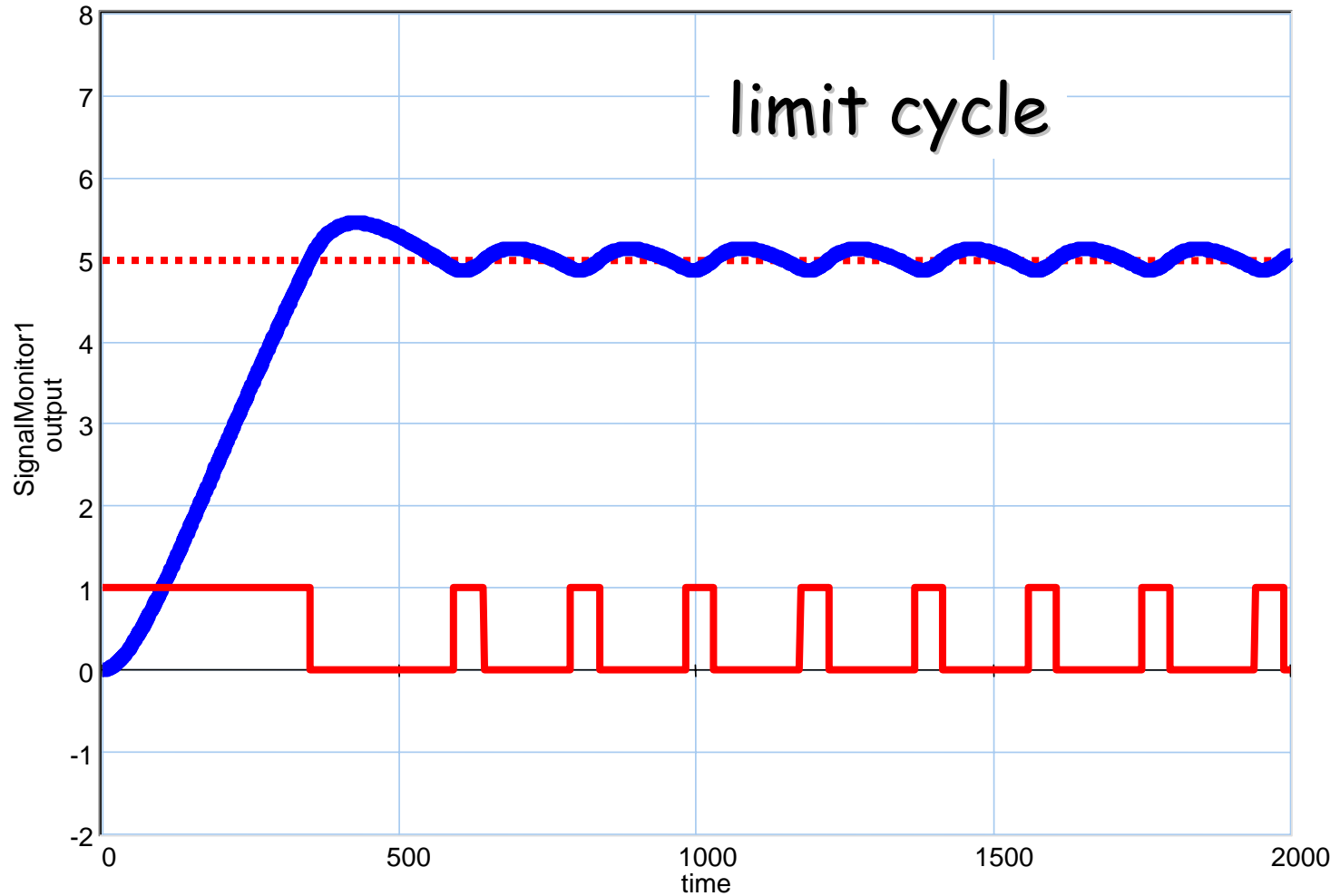


Relay plus hysteresis

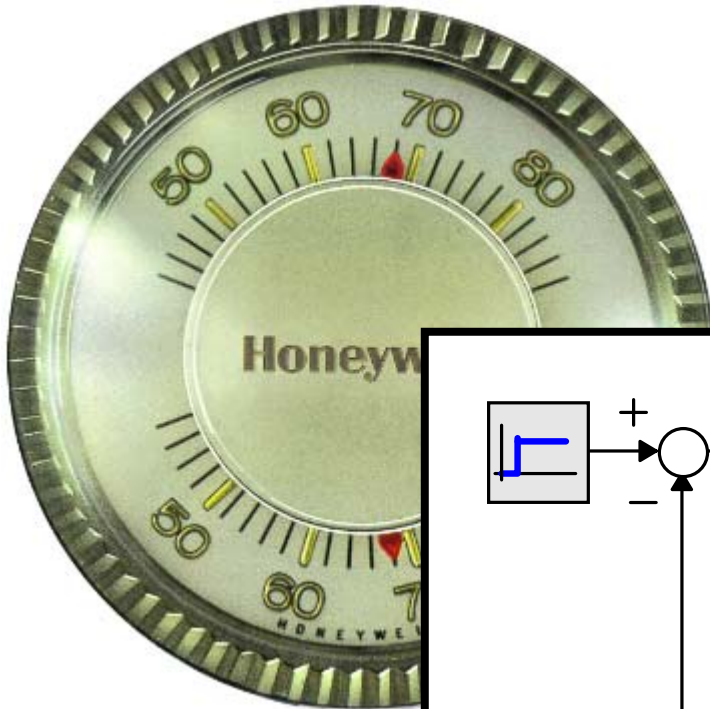


20-sim: Thermostat_demo3

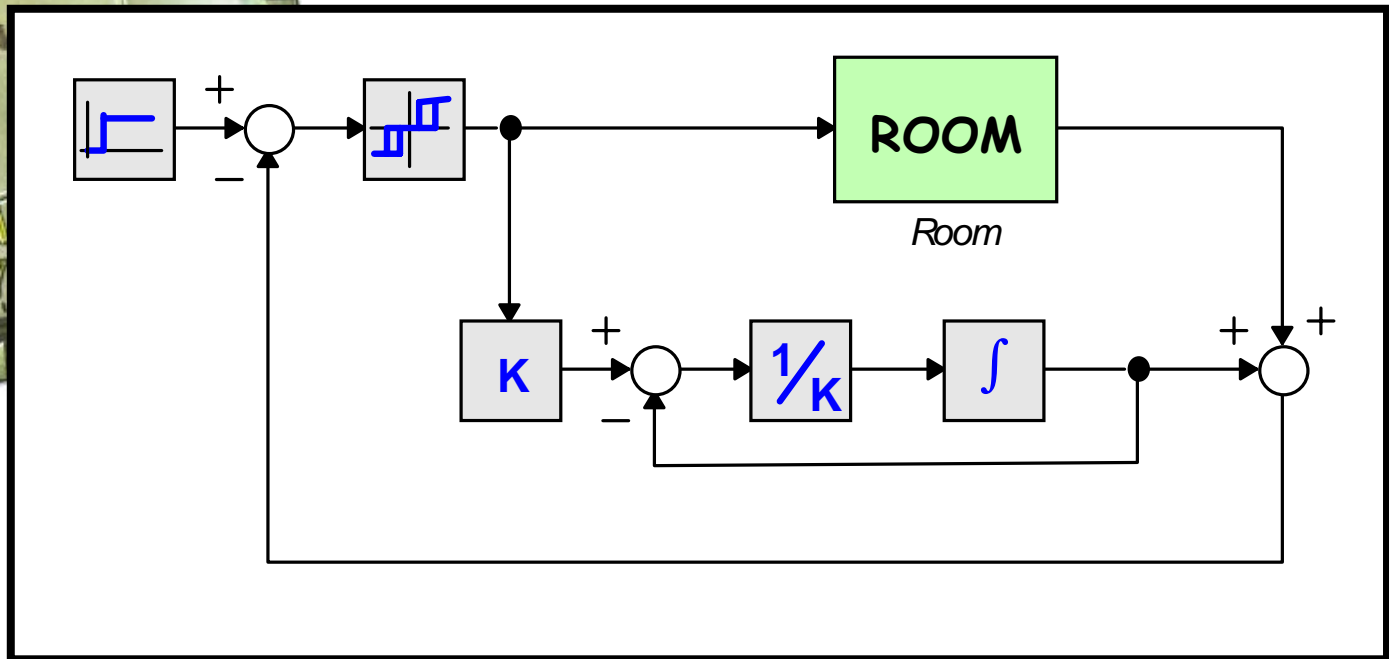
Relay plus hysteresis

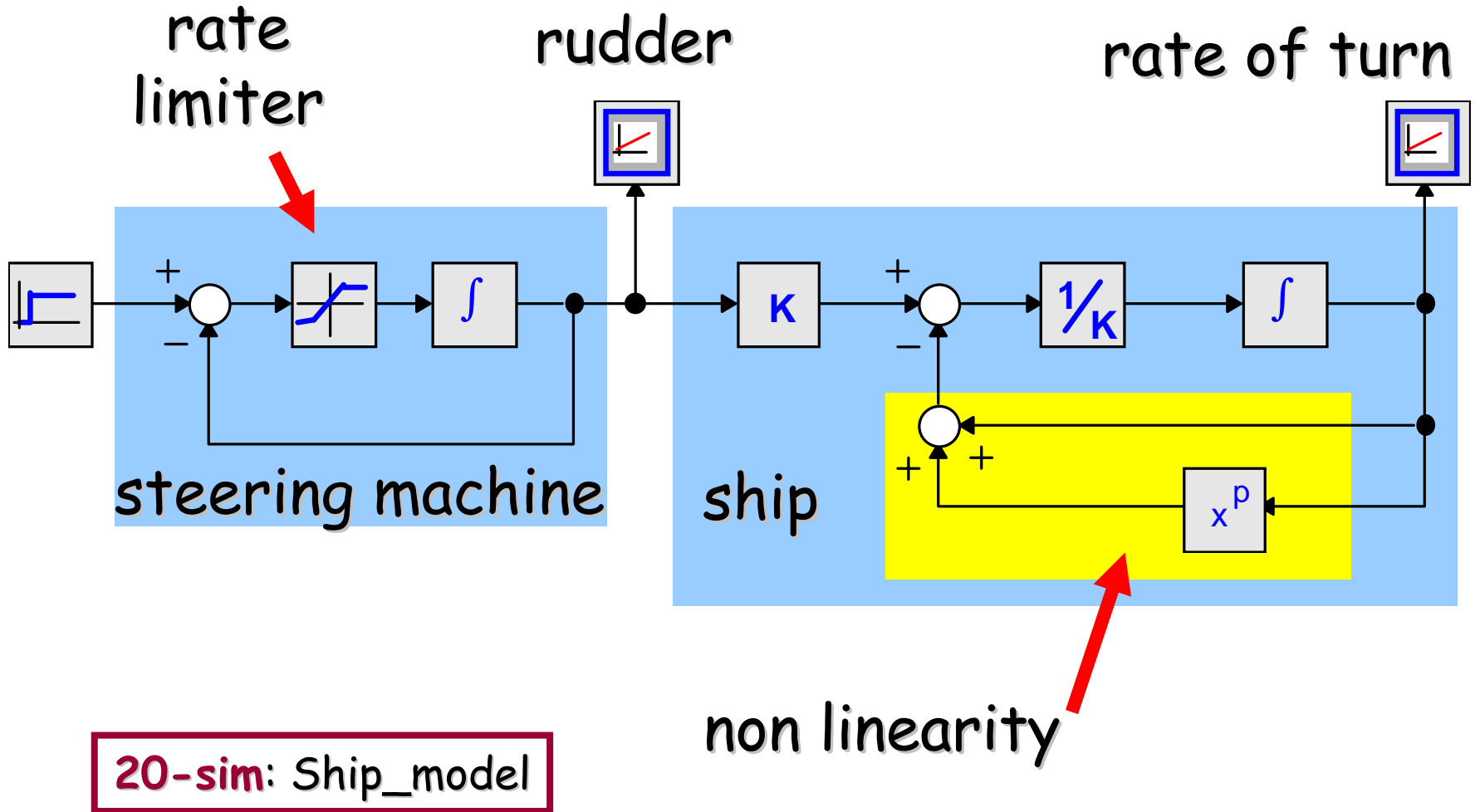


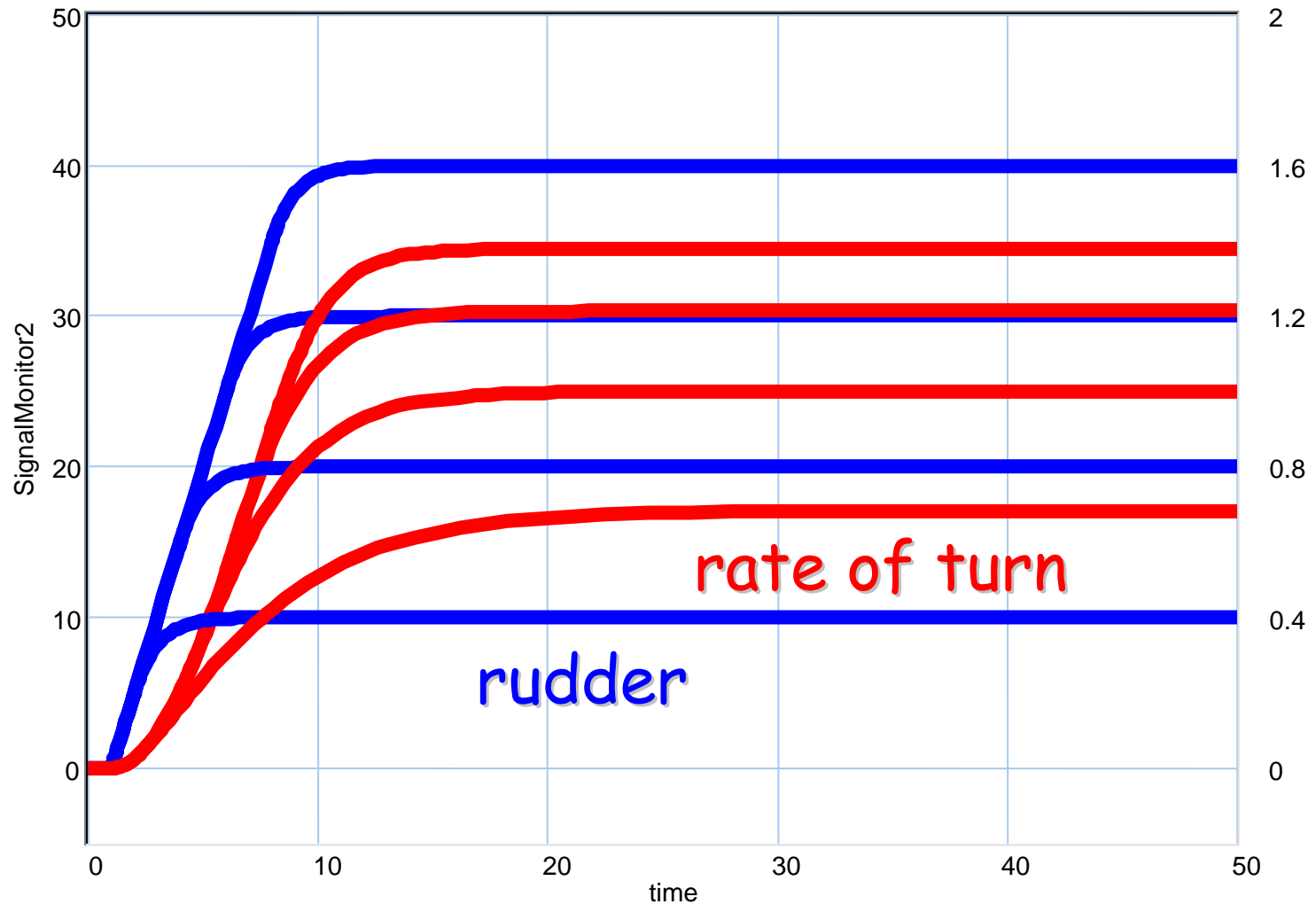
Room thermostat

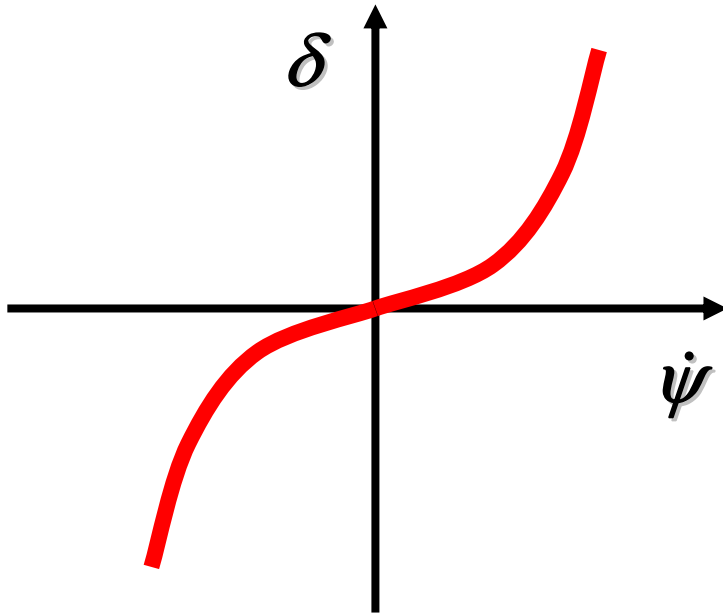


20-sim: CentralHeatingSystem



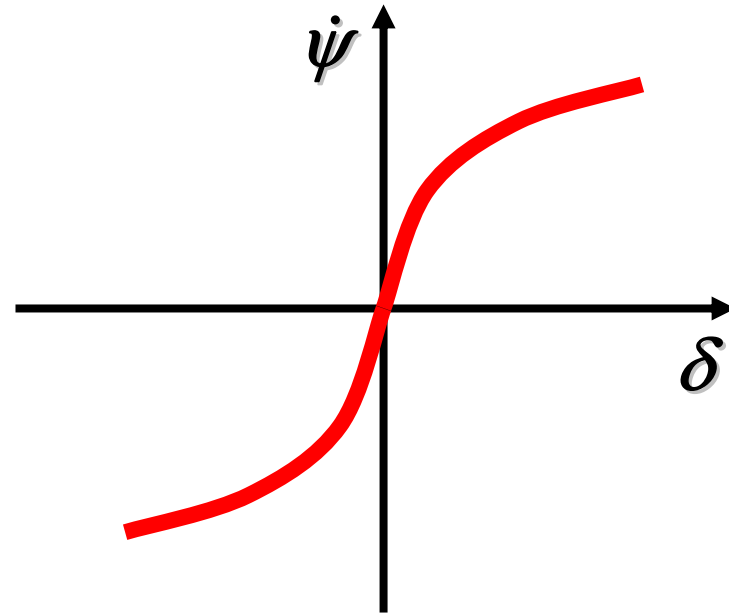




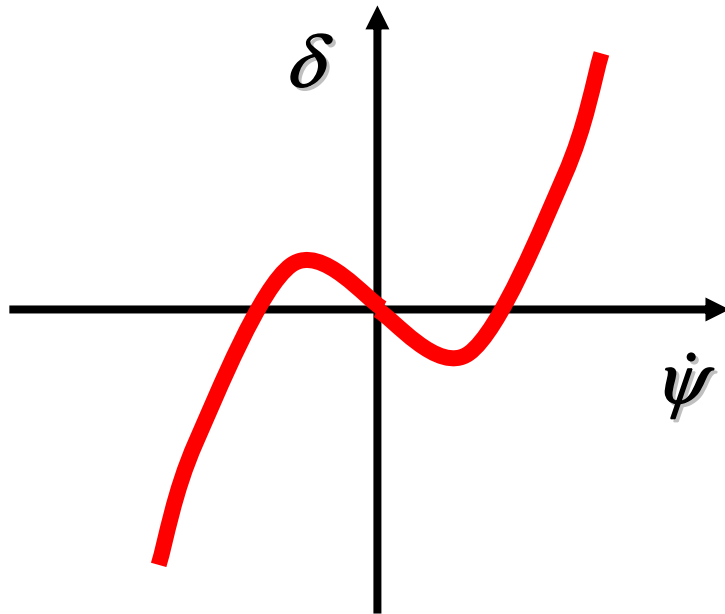


In feedback path:

$$\delta = c_1 \dot{\psi} + c_3 \dot{\psi}^3$$

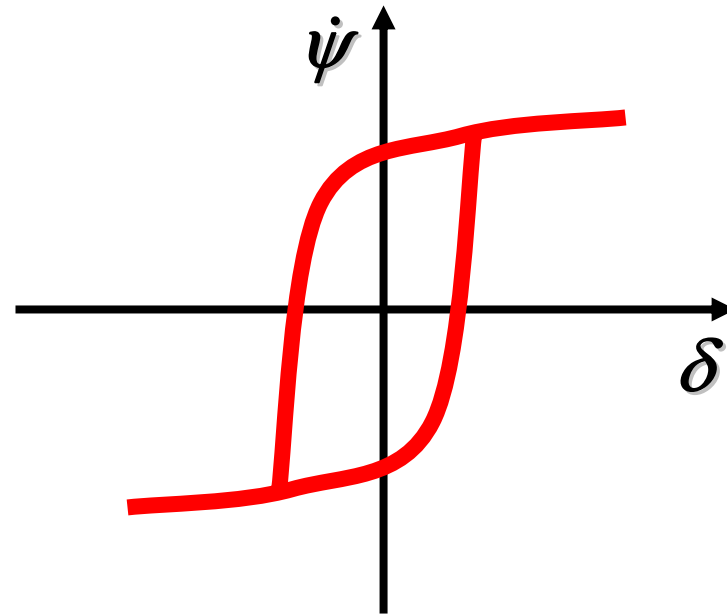


In forward path:

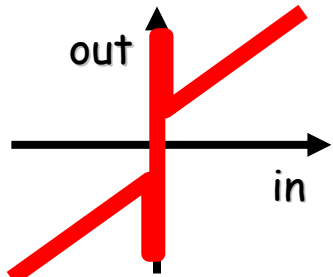


In feedback path:

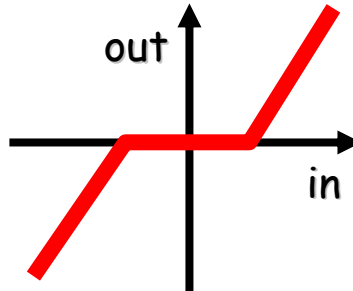
$$\delta = -c_1\psi + c_3\psi^3$$



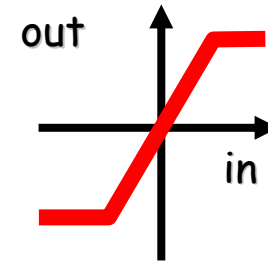
In forward path:



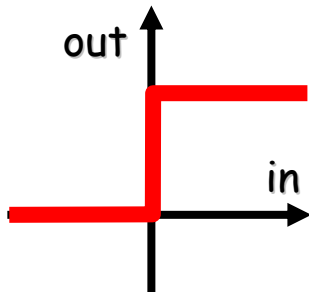
friction



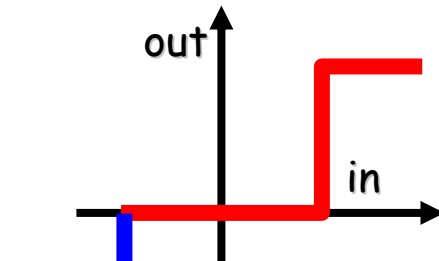
dead zone/band



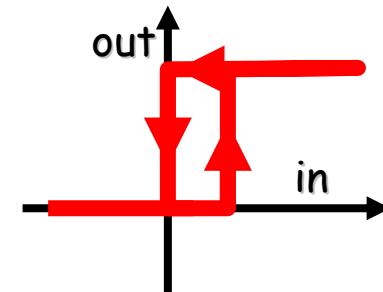
saturation



relay



relay with
dead zone

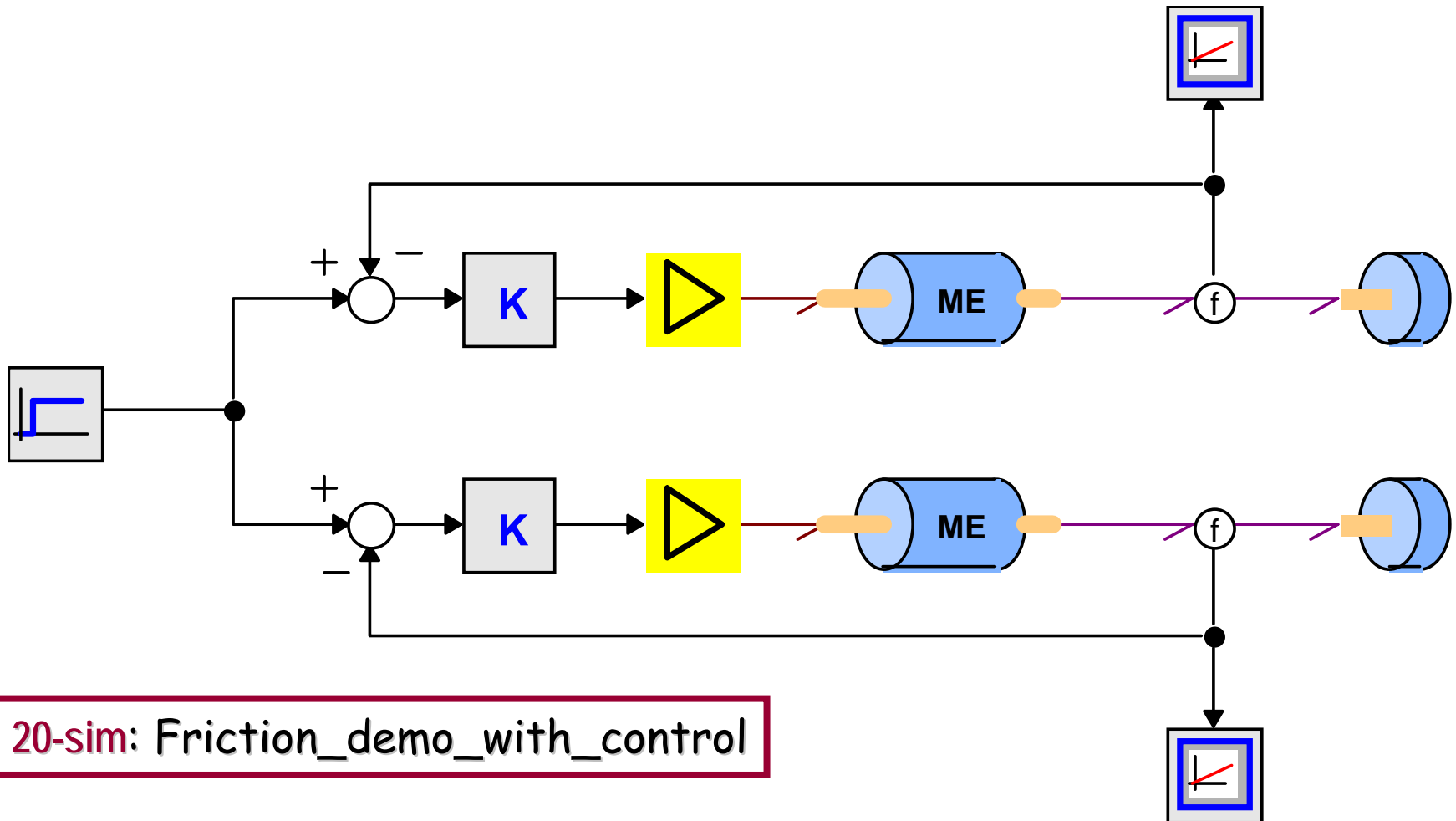


relay with
hysteresis

- Sometimes it is difficult or too expensive to make a linear system
 - friction, saturation
- Sometimes wanted
 - safety (maximum relay)
- It is Cheap
 - relay is a cheap power amplifier
e.g. thermostat, integrated with other functions: sensing, setpoint

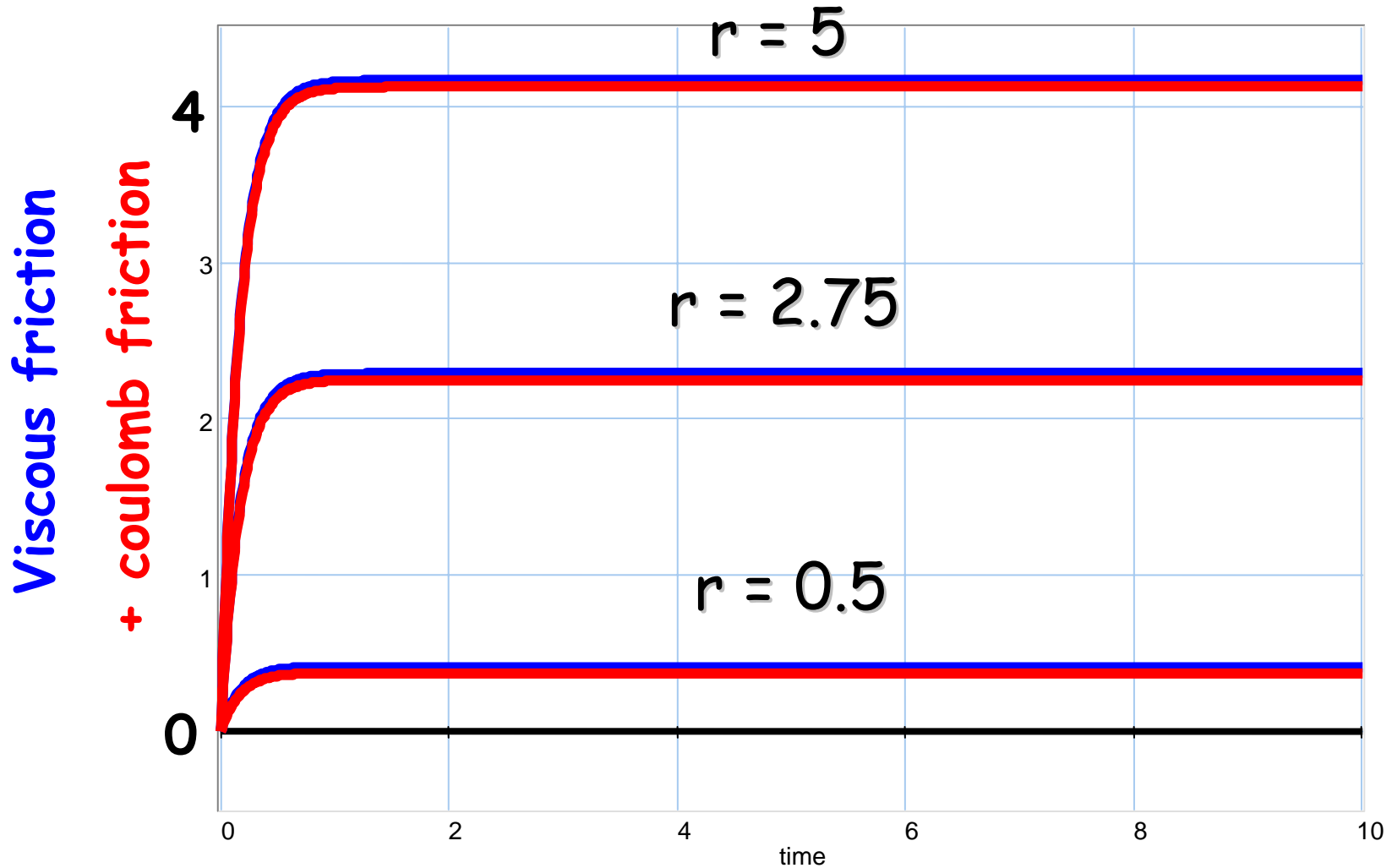
- Behaviour of a good feedback system is only determined by the elements in the feedback path.
- High gain feedback can compensate for unwanted non-linearities, such as friction

Tacho feedback

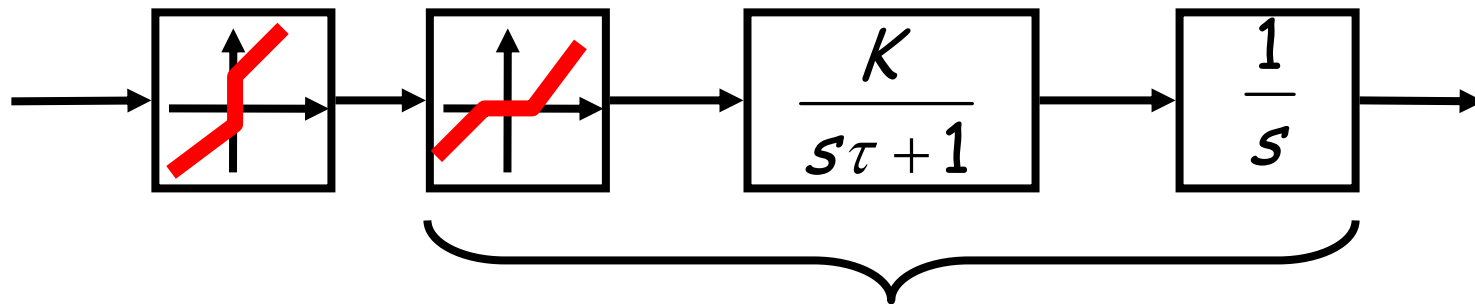


20-sim: Friction_demo_with_control

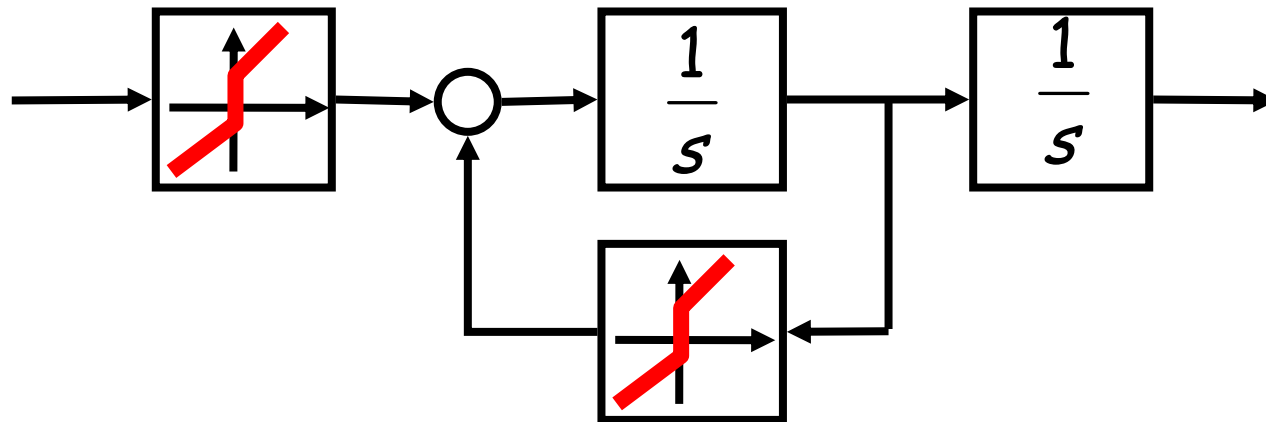
Influence of non linearities with tacho feedback



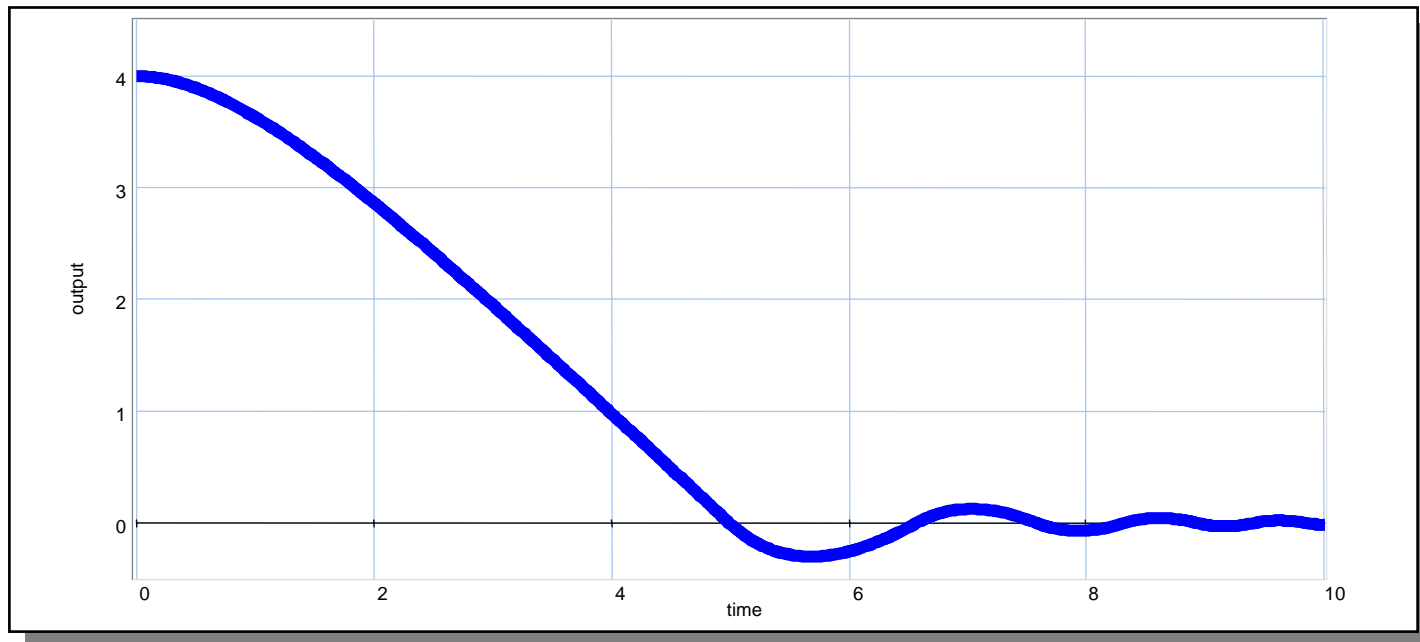
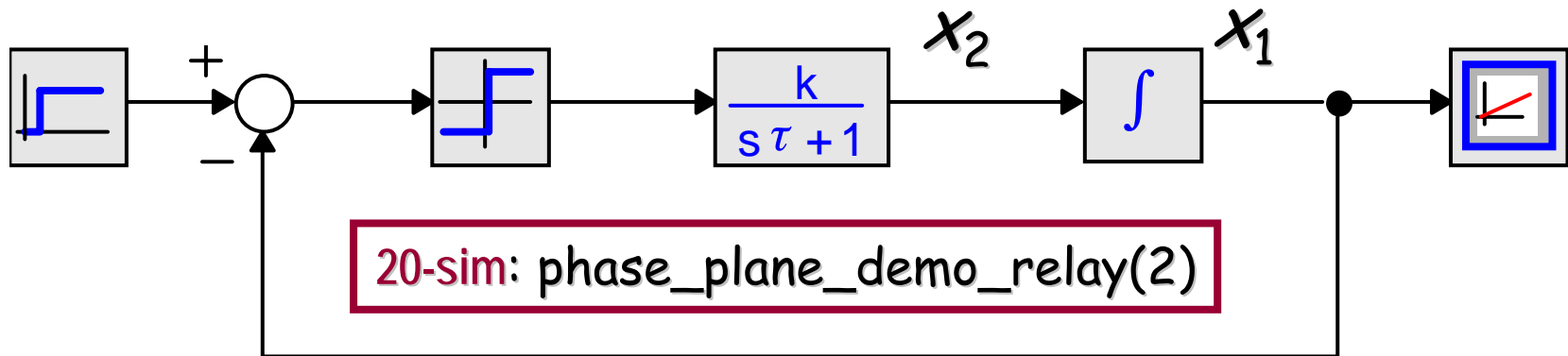
- Feedback can considerably reduce the influence of non linearities
- Especially tacho feedback is an effective means to get rid of the deteriorating effects of static and coulomb friction
- Other possibility:
inverse characteristic in series
 - much more sensitive for parameters



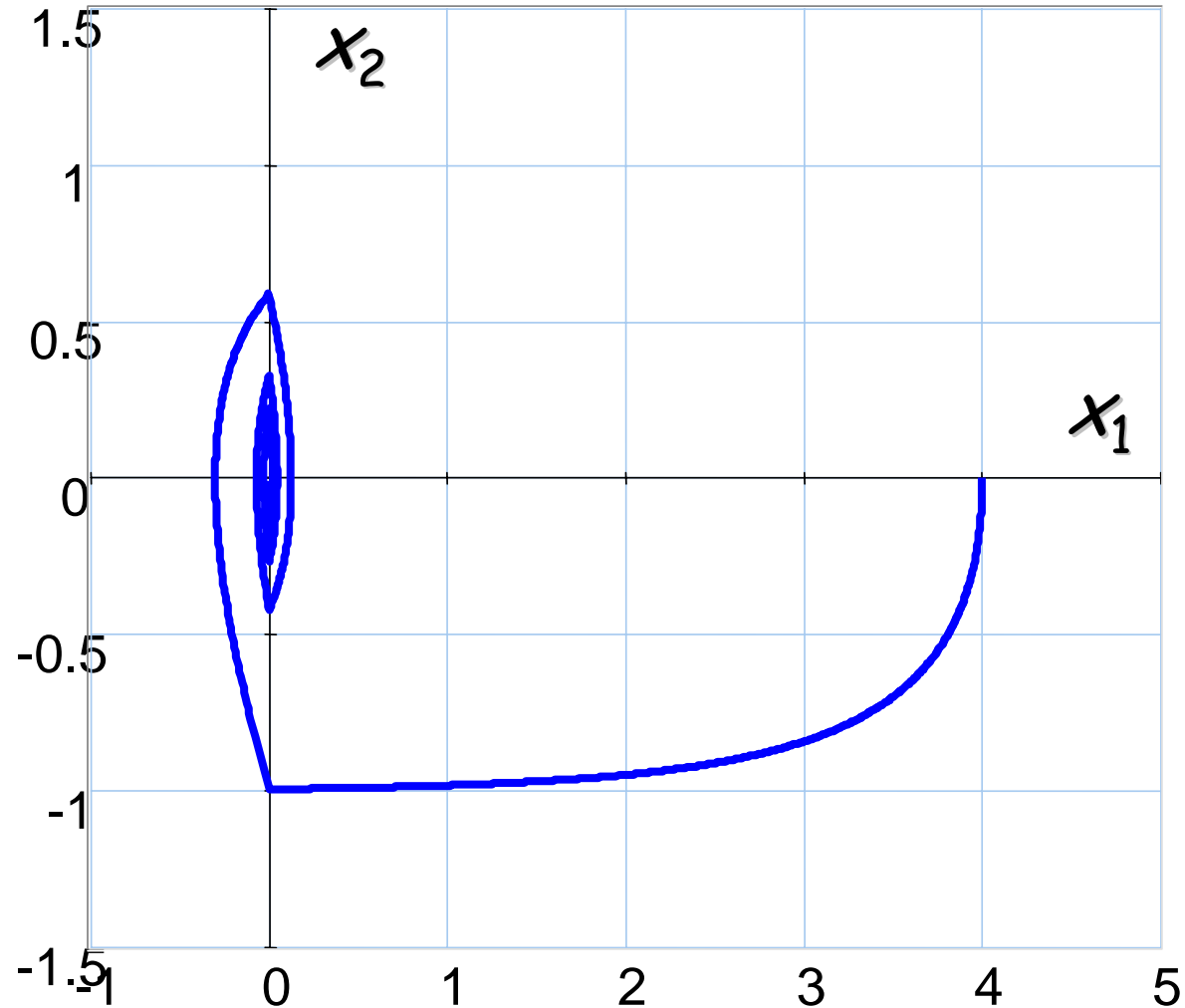
motor with friction

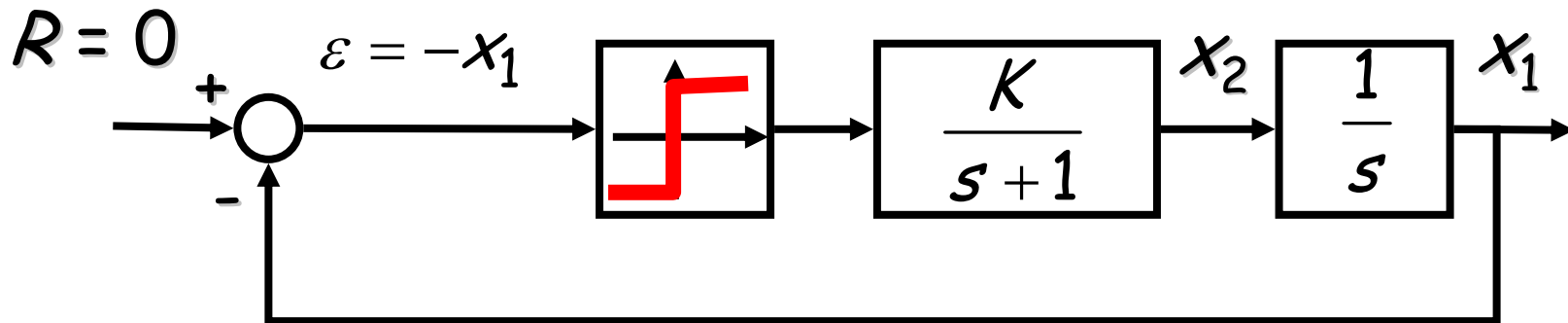


Phase plane analysis



Phase plane plot





Relay output: $\pm M$

$$\dot{x}_1 = x_2$$

$$K = 1$$

$$\dot{x}_2 = -x_2 + M \quad \text{if } \varepsilon > 0 \quad (x_1 < 0)$$

$$\dot{x}_2 = -x_2 - M \quad \text{if } \varepsilon < 0 \quad (x_1 > 0)$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_2 + M \quad \text{if } \varepsilon > 0 \quad (x_1 < 0)$$

$$\dot{x}_2 = -x_2 - M \quad \text{if } \varepsilon < 0 \quad (x_1 > 0)$$

isocline

$$\frac{dx_2}{dx_1} = \frac{\dot{x}_2}{\dot{x}_1} = \frac{-x_2 + M}{x_2} \quad (x_1 < 0)$$

$$\frac{dx_2}{dx_1} = \frac{\dot{x}_2}{\dot{x}_1} = \frac{-x_2 - M}{x_2} \quad (x_1 > 0)$$

$$\frac{dx_2}{dx_1} = -1 + \frac{M}{x_2} = m$$

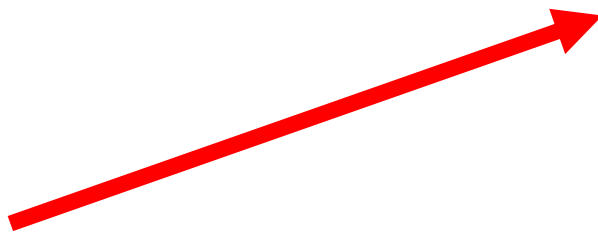
$$\frac{dx_2}{dx_1} = -1 - \frac{M}{x_2} = m$$

Some values

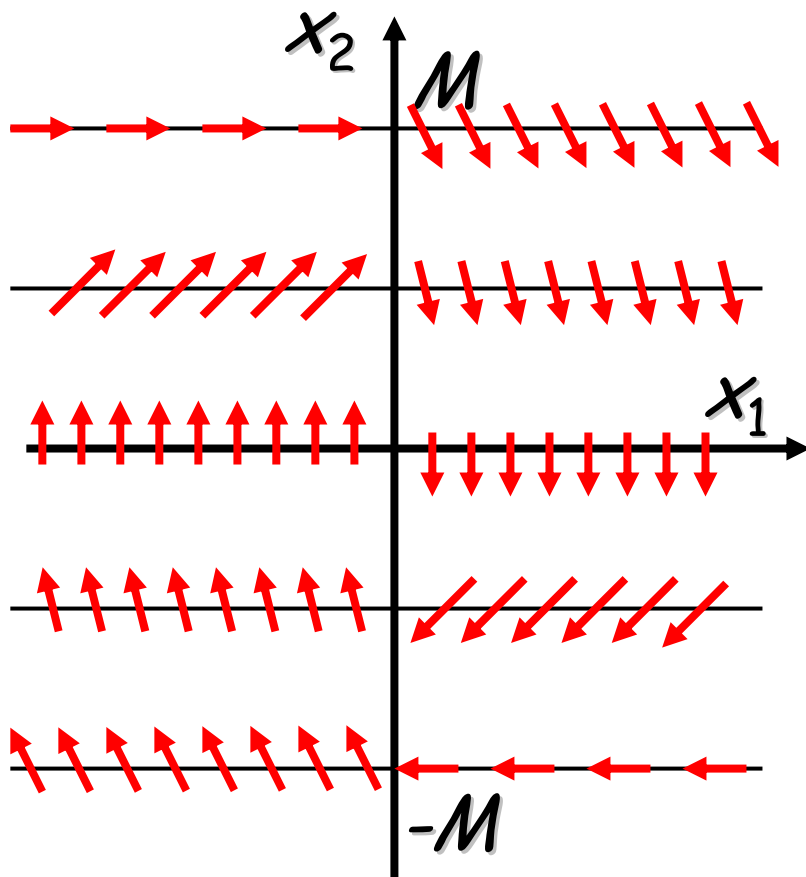
$$m = -1 + \frac{M}{x_2} \quad (x_1 < 0)$$

$$m = -1 - \frac{M}{x_2} \quad (x_1 > 0)$$

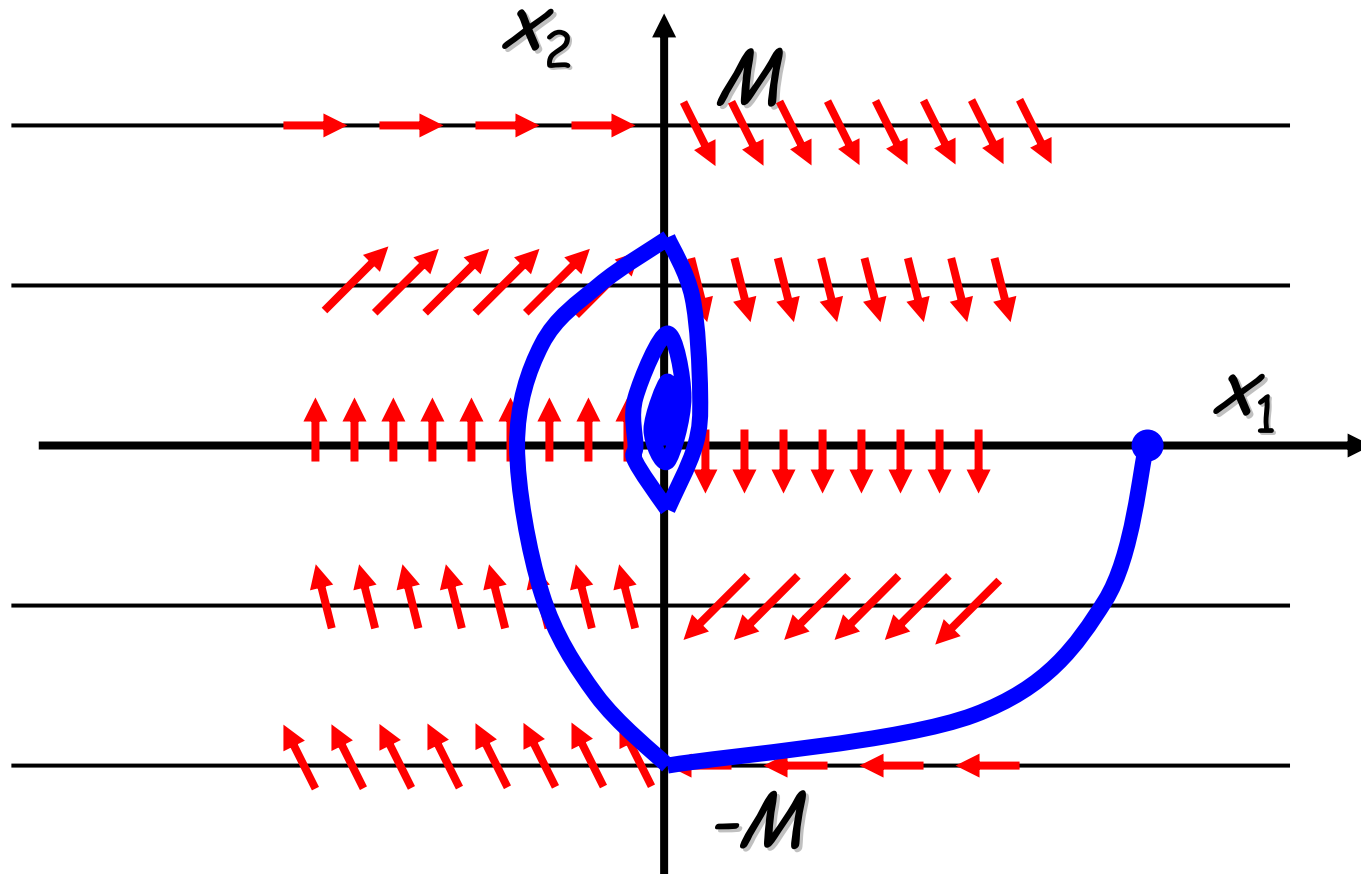
m



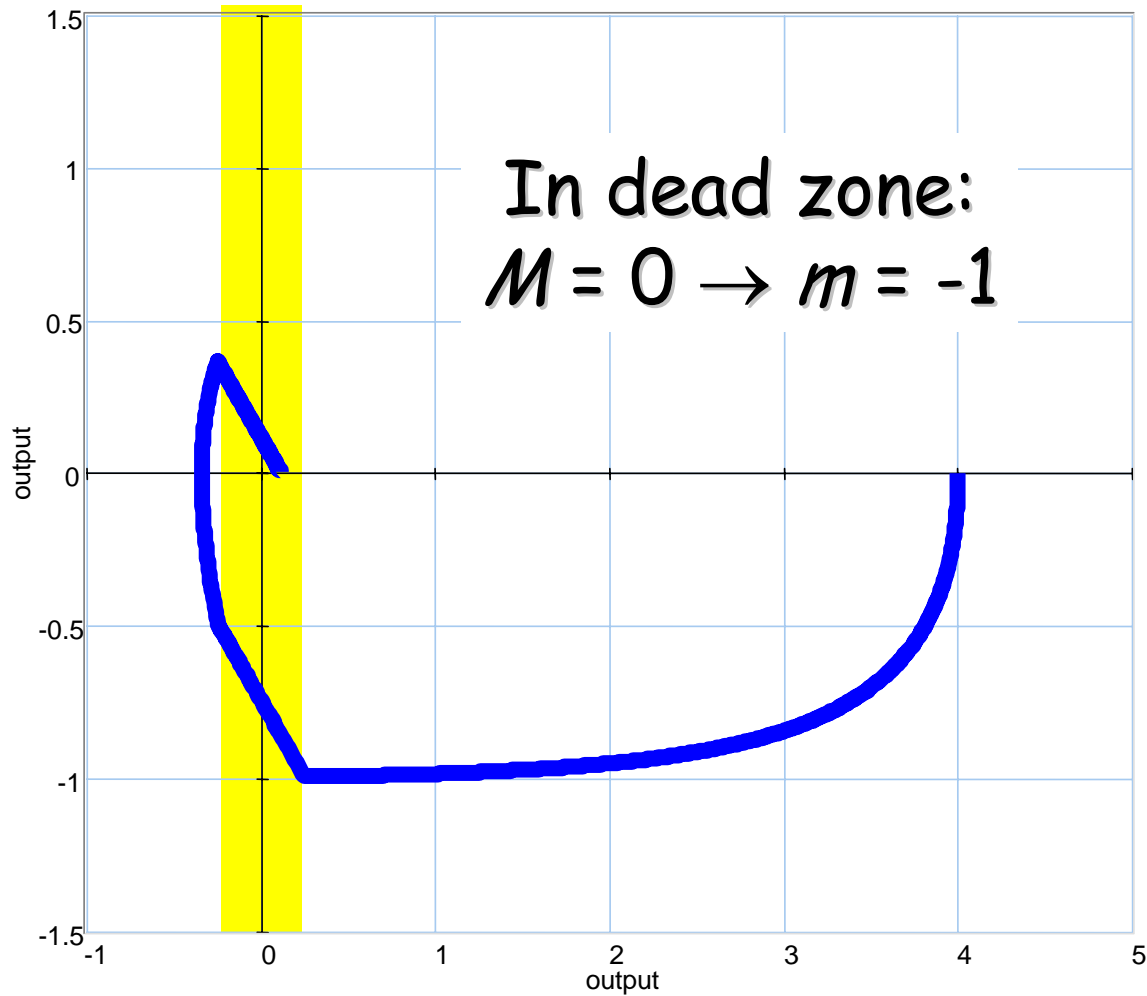
<u>x_2</u>	<u>$x_1 < 0$</u>	<u>$x_1 > 0$</u>
0	∞	$-\infty$
M	0	-2
$-M$	-2	0
$M/2$	1	-3
<u>$-M/2$</u>	<u>-3</u>	<u>1</u>



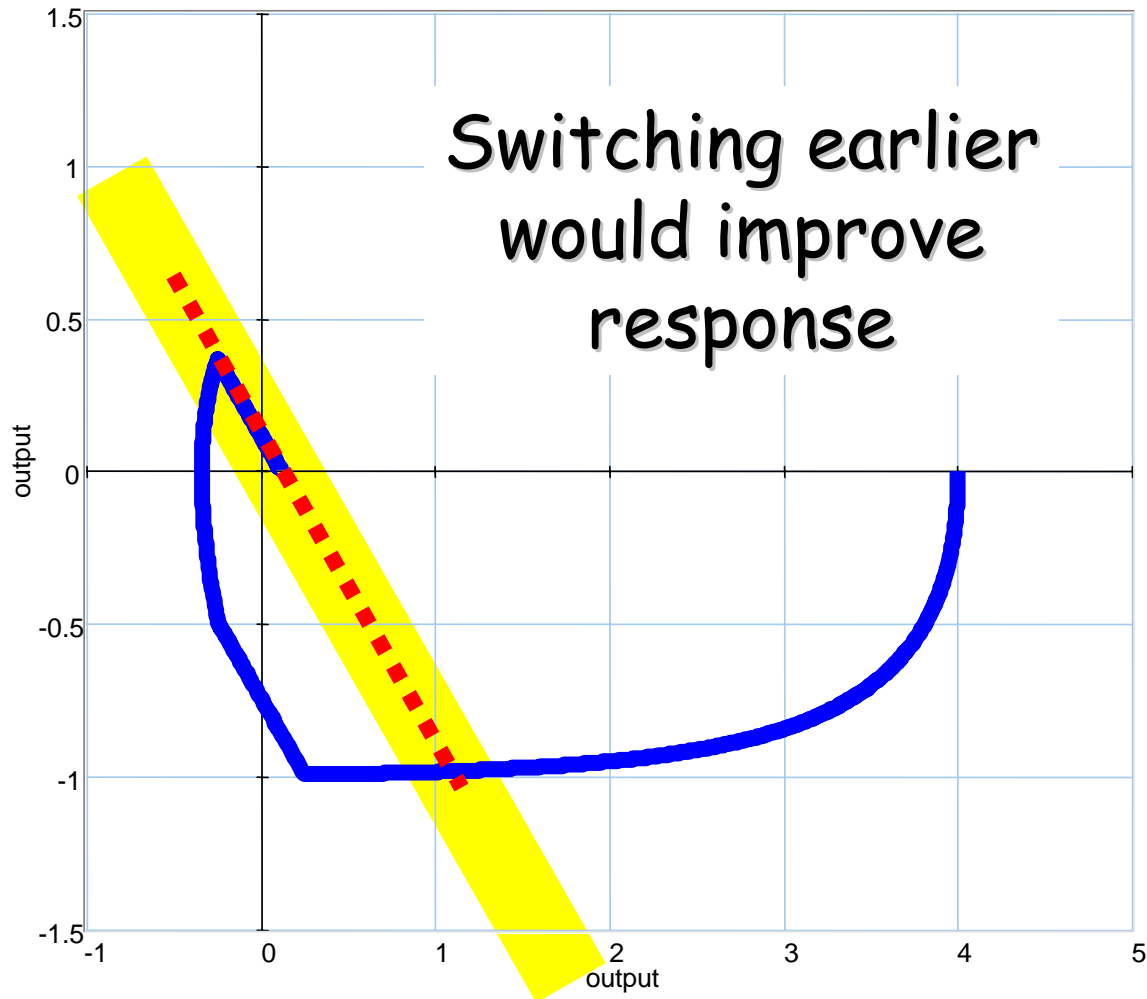
x_2	$x_1 < 0$	$x_1 > 0$
0	∞	$-\infty$
M	0	-2
$-M$	-2	0
$M/2$	1	-3
$-M/2$	-3	1

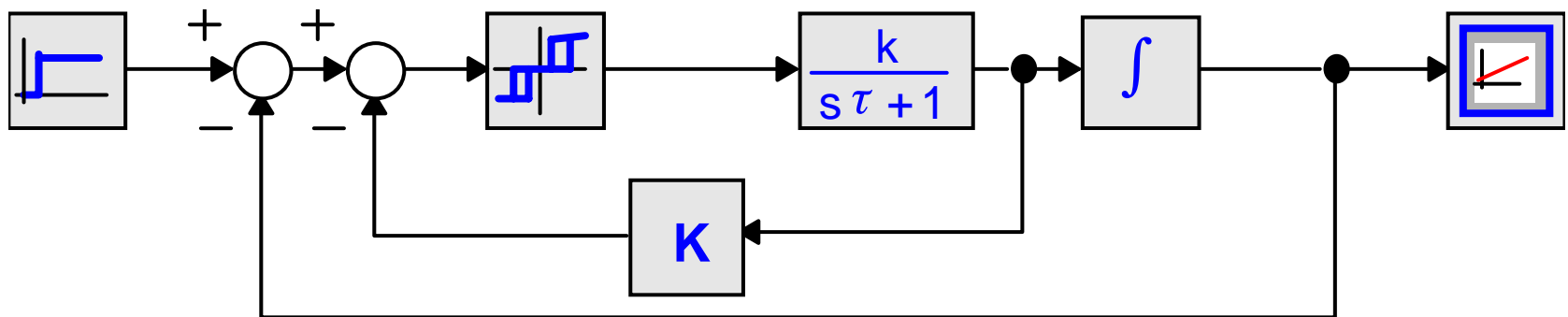


Relay with dead zone



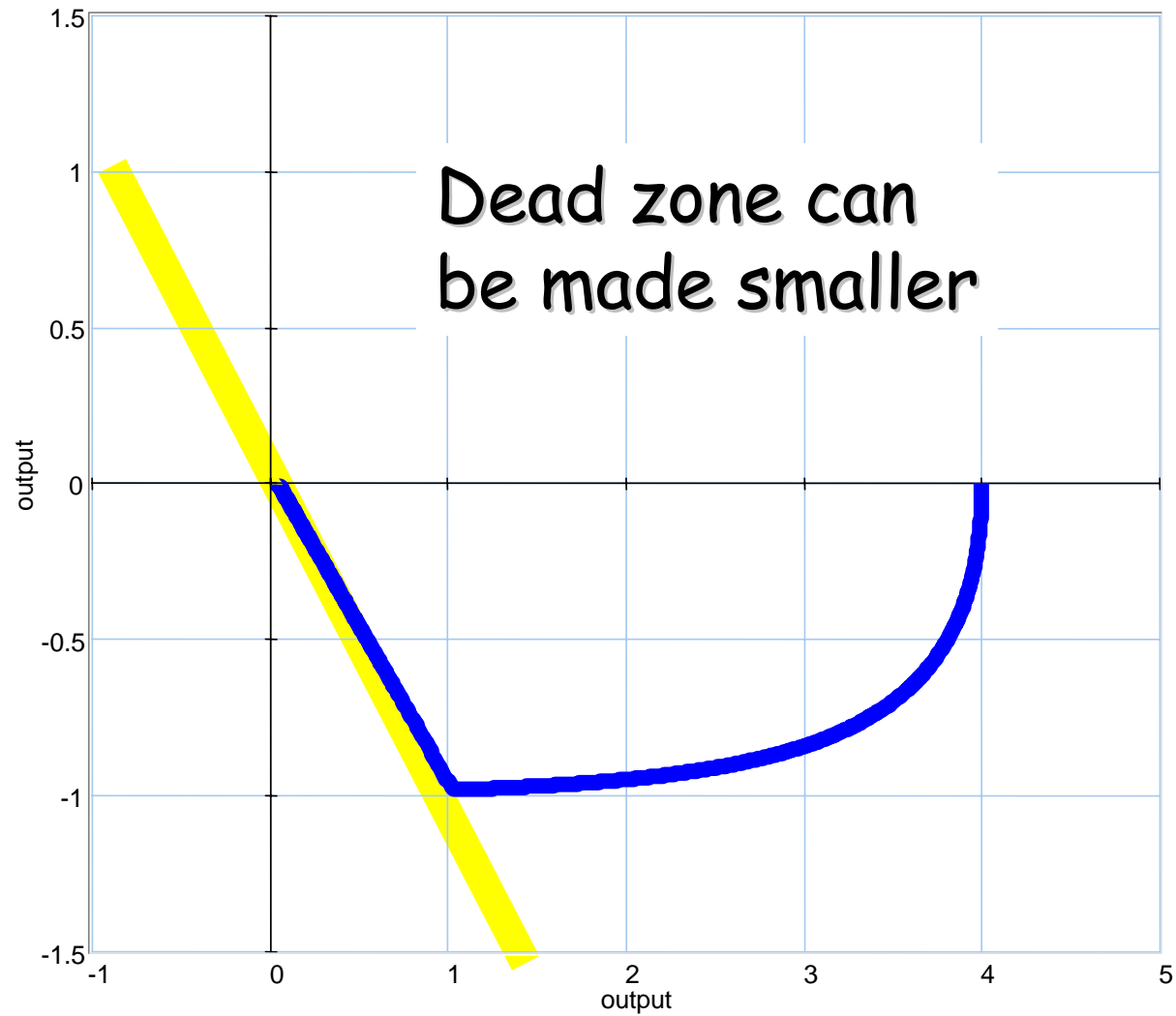
20-sim: Phase_plane_demo_relay_DZ

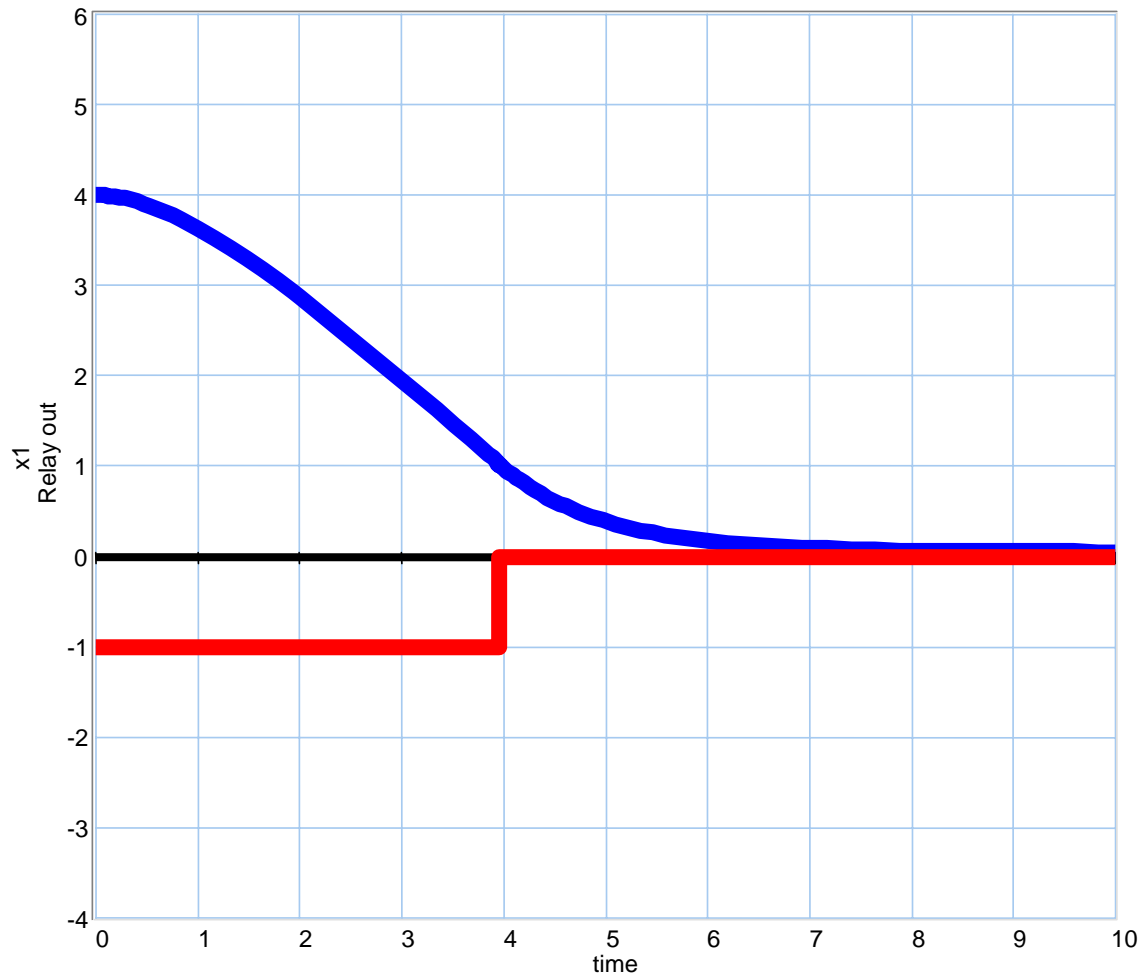




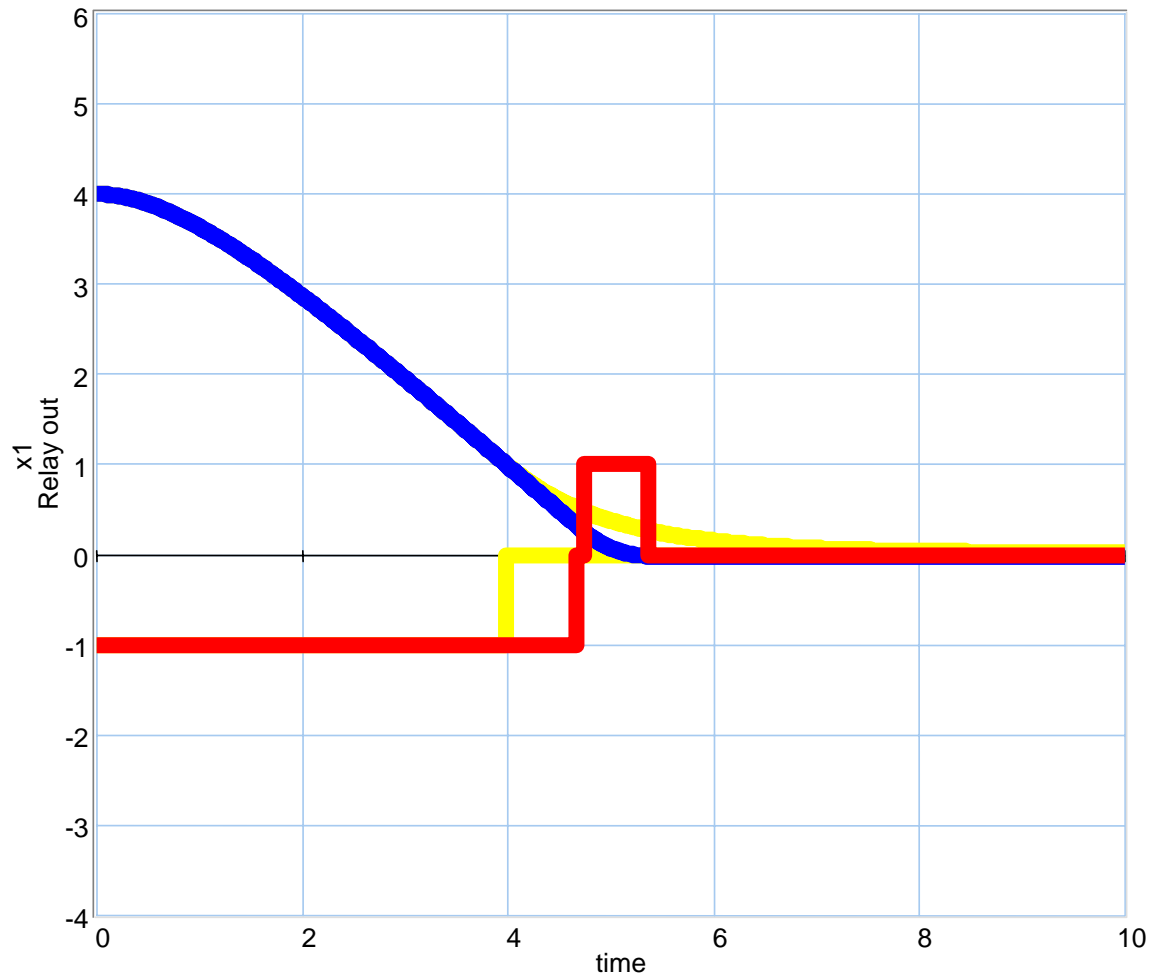
Phaseplane_demo_relay_DZ_switchline

```
+ import exp:  
Phaseplane_demo_relay_DZ_switchline bang bang
```

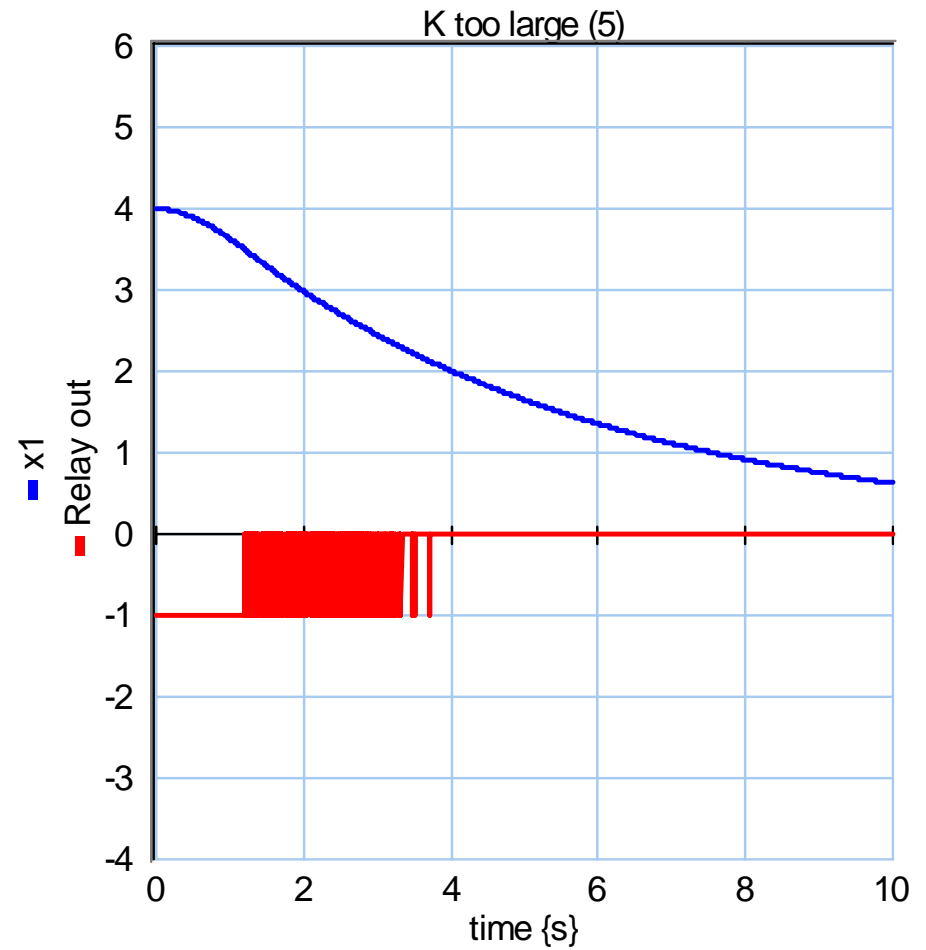
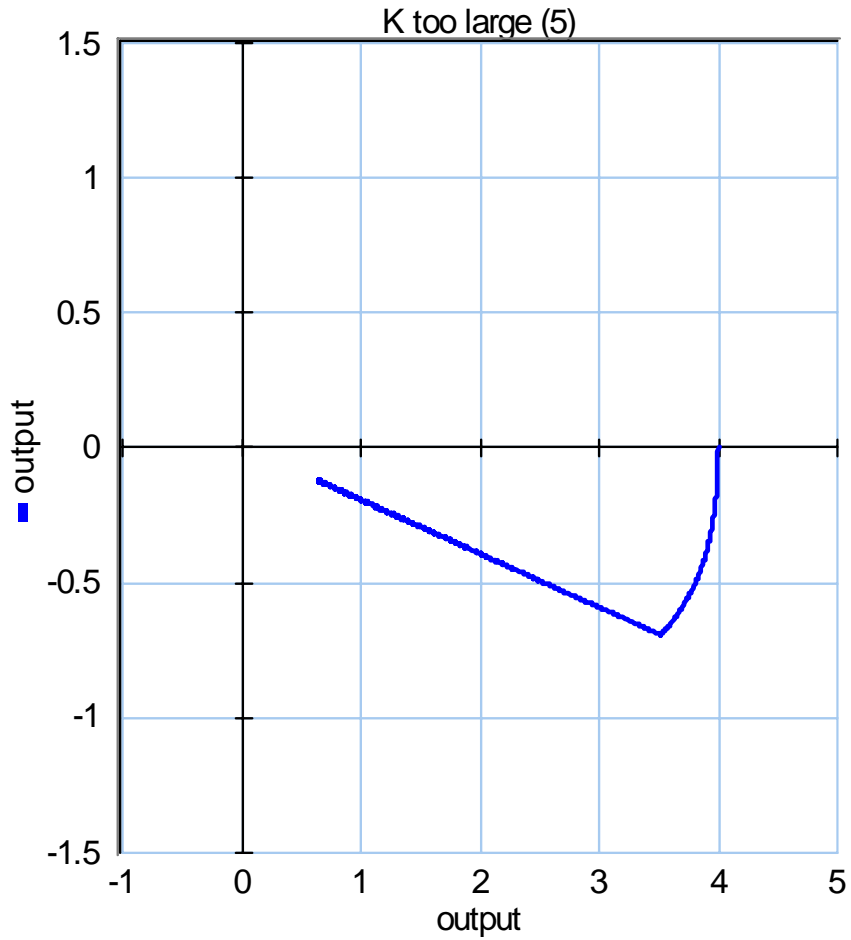




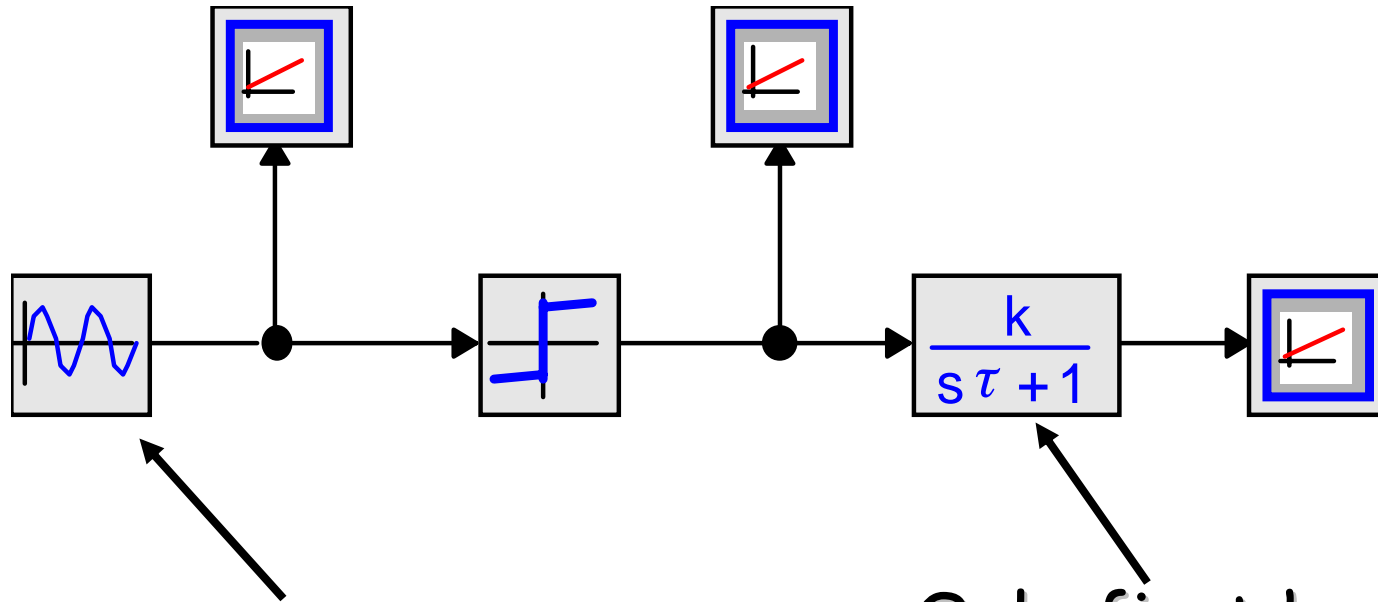
Time optimal (bang bang)



K_d too large

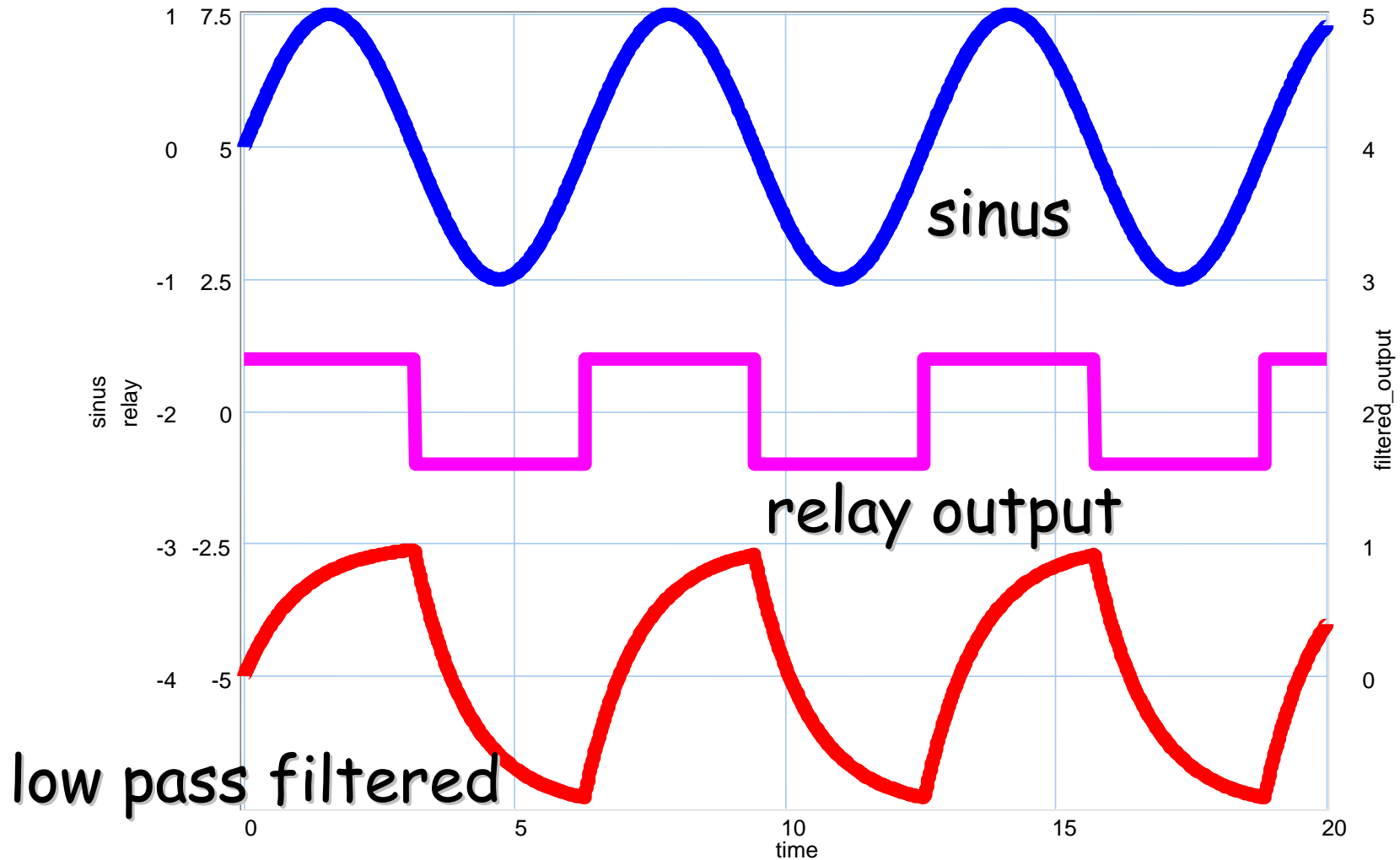


Describing functions



We consider systems at the border of instability: i.e. sinusoidal signals

Only first harmonic of the signal, because of low-pass character of process



Consider a NL-element with output signal $y(t)$ and $x(t) = x_{\max} \sin(\omega t)$ as input signal

Fourier series:

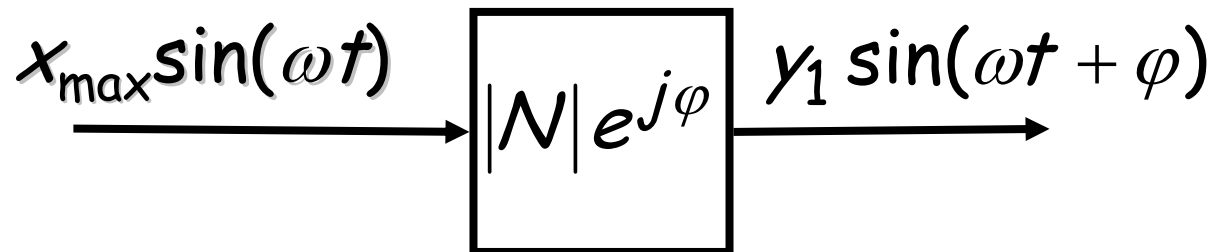
$$y(t) \approx y_1 \sin(\omega t + \varphi) = a_1 \cos(\omega t) + b_1 \sin(\omega t)$$

$$a_1 = \frac{1}{\pi} \int_0^{2\pi} y \cos(\omega t) d(\omega t) \quad y_1 = \sqrt{a_1^2 + b_1^2}$$

$$b_1 = \frac{1}{\pi} \int_0^{2\pi} y \sin(\omega t) d(\omega t) \quad \varphi = \arctan\left(\frac{a_1}{b_1}\right)$$

The describing function of the NL-element, N , is:

$$N = |N| e^{j\varphi} = \frac{y_1}{x_{\max}} e^{j\varphi}$$



$$y(t) = M \quad \text{if } 0 < \omega t < \pi$$

$$y(t) = -M \quad \text{if } \pi < \omega t < 2\pi$$

$$a_1 = \frac{1}{\pi} \int_0^{2\pi} y \cos(\omega t) d(\omega t) =$$

$$= \frac{1}{\pi} \int_0^{\pi} M \cos(\omega t) d(\omega t) + \frac{1}{\pi} \int_{\pi}^{2\pi} -M \cos(\omega t) d(\omega t)$$

$$a_1 = \frac{1}{\pi} (M \sin(\omega t)) \Big|_0^{\pi} + \frac{1}{\pi} (-M \sin(\omega t)) \Big|_{\pi}^{2\pi} = 0$$

$$\varphi = 0$$

$$\varphi = \arctan\left(\frac{a_1}{b_1}\right)$$

For all radial symmetric, single valued non linearities, $a_1 = 0$.

This implies that $\varphi = 0$ (no phase lag)

$$y(t) = M \text{ if } 0 < \omega t < \pi, \quad y(t) = -M \text{ if } \pi < \omega t < 2\pi$$

$$b_1 = \frac{1}{\pi} \int_0^{2\pi} y \sin(\omega t) d(\omega t) =$$

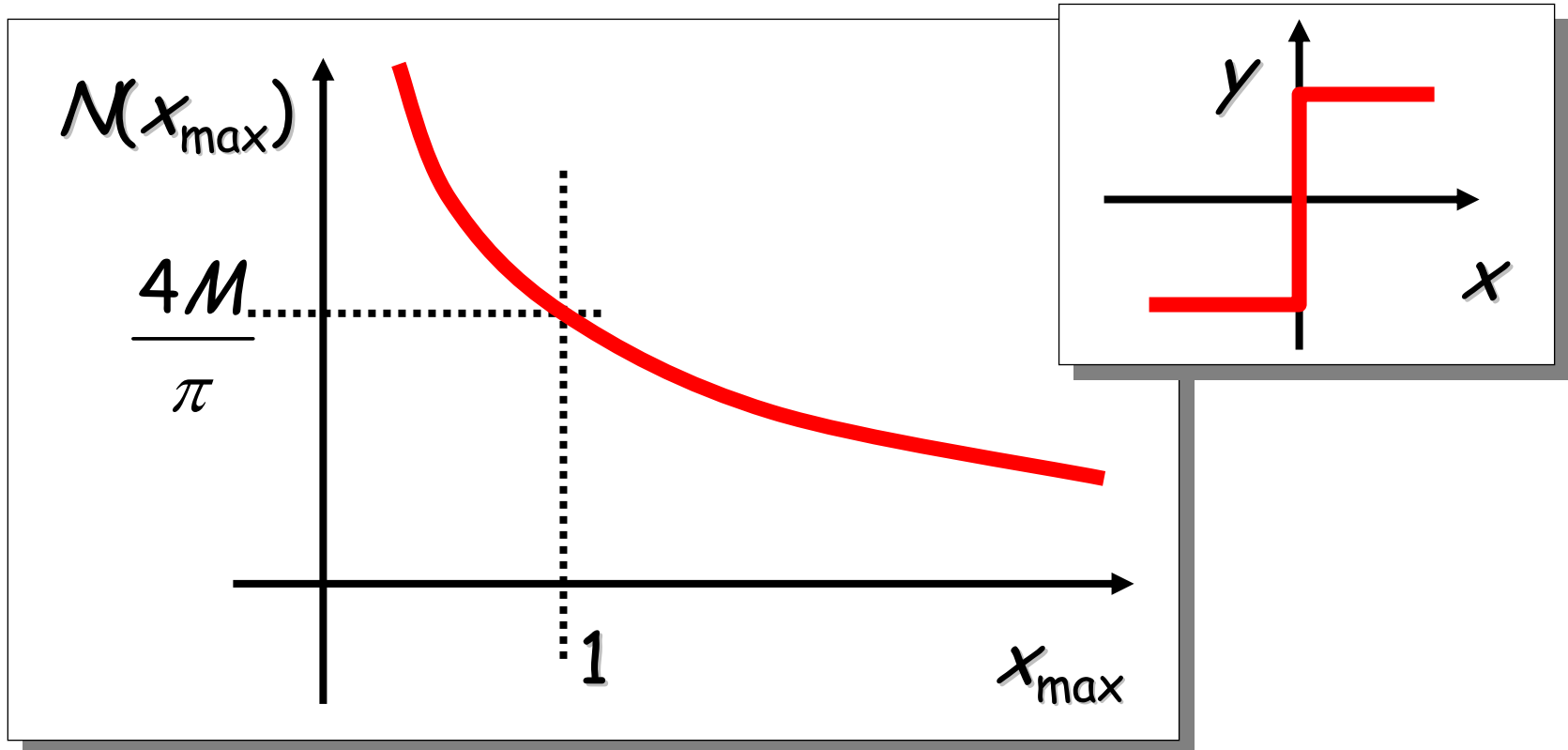
$$= \frac{1}{\pi} \int_0^{\pi} M \sin(\omega t) d(\omega t) + \frac{1}{\pi} \int_{\pi}^{2\pi} -M \sin(\omega t) d(\omega t)$$

$$b_1 = \frac{1}{\pi} (-M \cos(\omega t)) \Big|_0^{\pi} + \frac{1}{\pi} - (-M \sin(\omega t)) \Big|_{\pi}^{2\pi} =$$

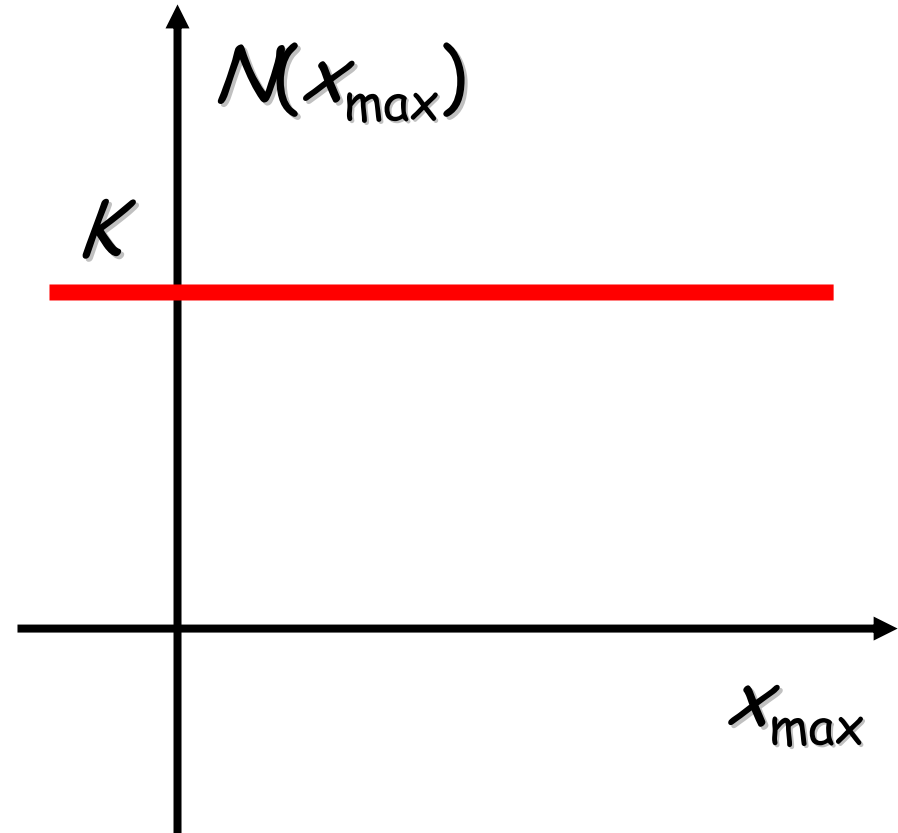
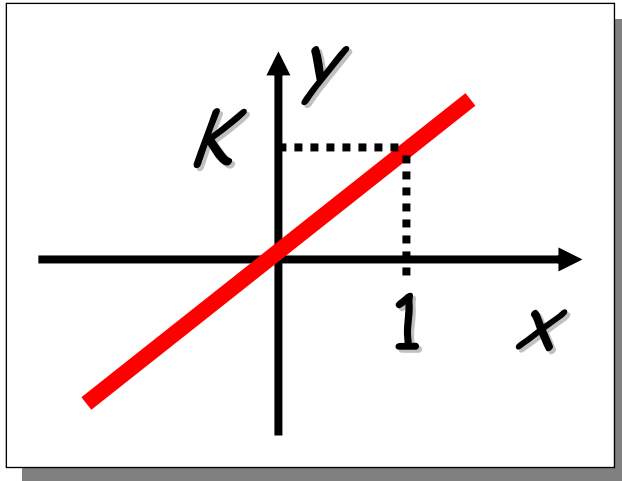
$$= \frac{1}{\pi} [-(-M - M)] + \frac{1}{\pi} [-(-M - M)] = \frac{4M}{\pi}$$

Describing function of relay

$$N = \frac{y_1}{x_{\max}} = \frac{b_1}{x_{\max}} = \frac{4M}{\pi} \frac{1}{x_{\max}}$$

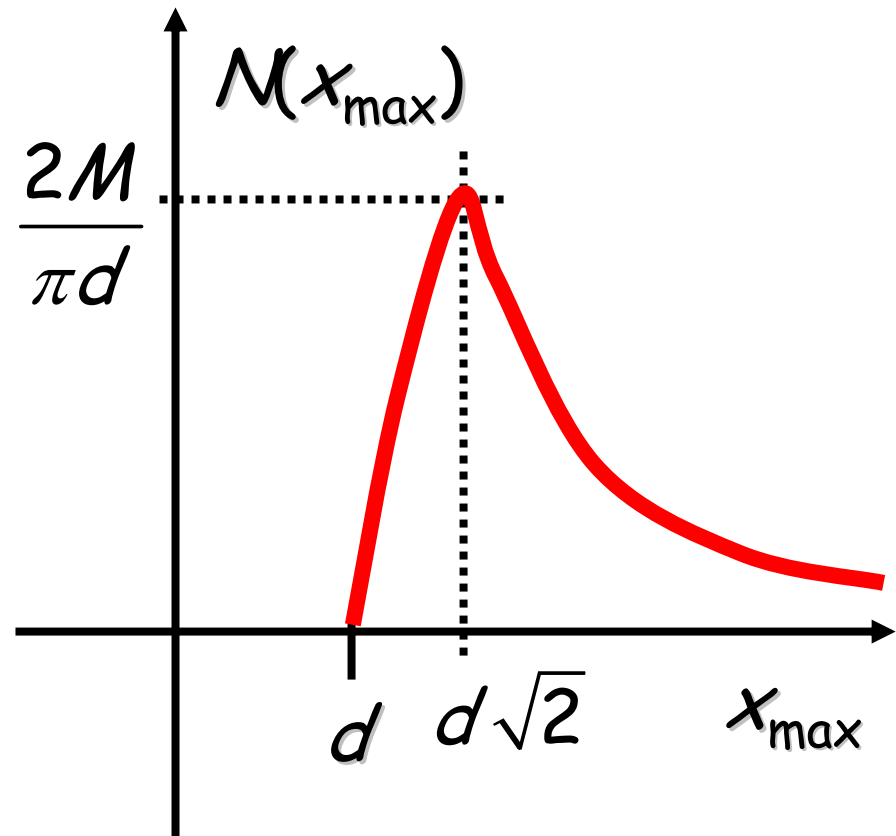
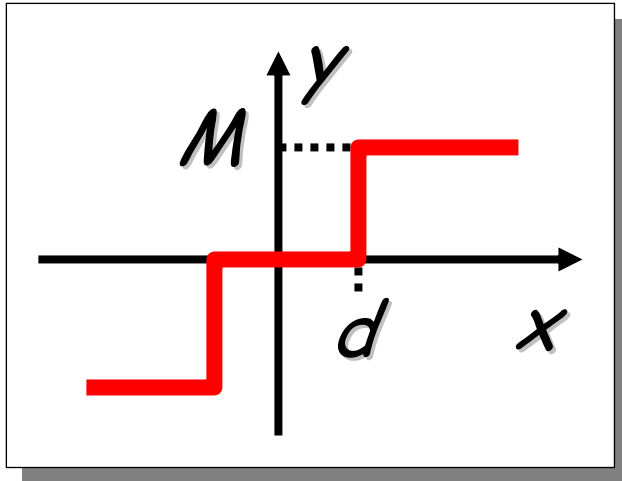


Describing function (linear gain)

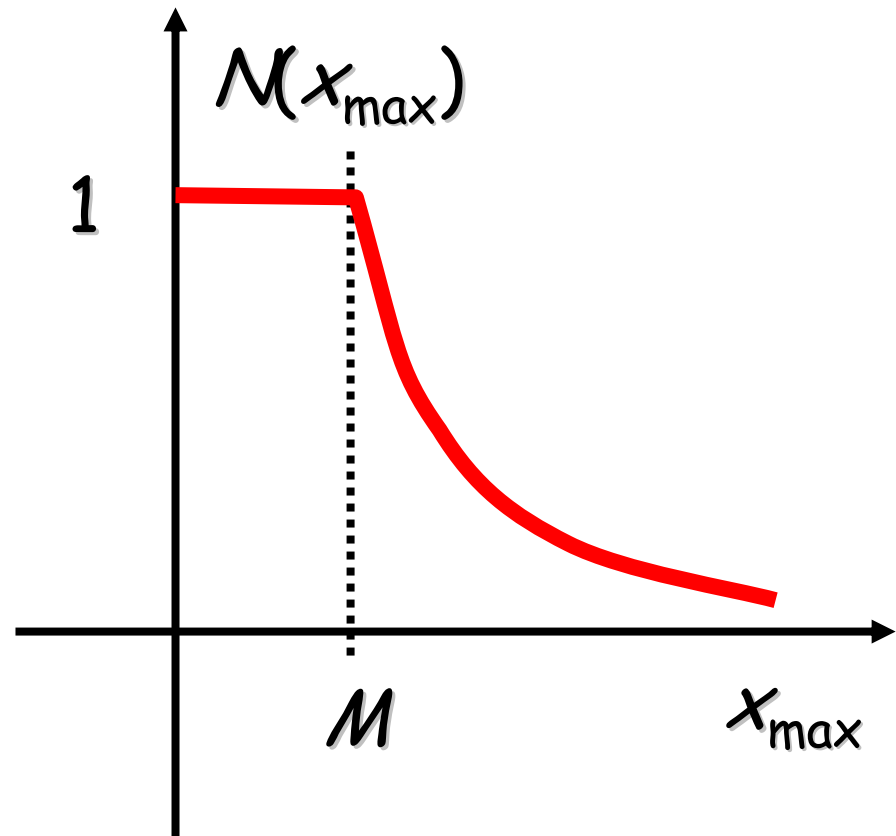
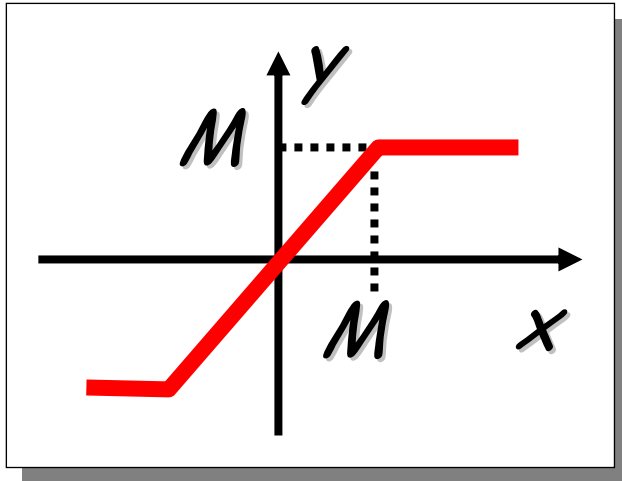


DF is a kind
of amplitude-
dependent gain

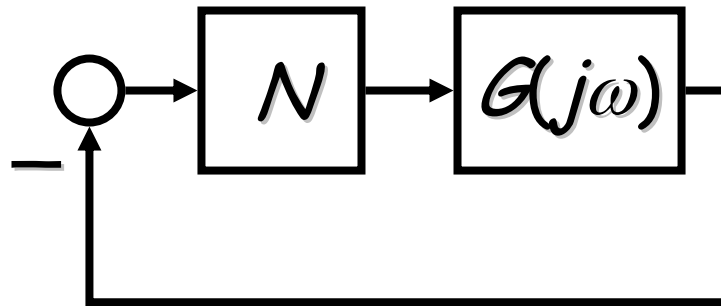
Describing function (relay with dead zone)



Describing function (saturation)



- Describing function should only be used for sinusoidal signals
 - systems on the border of instability



Close loop system (characteristic equation)

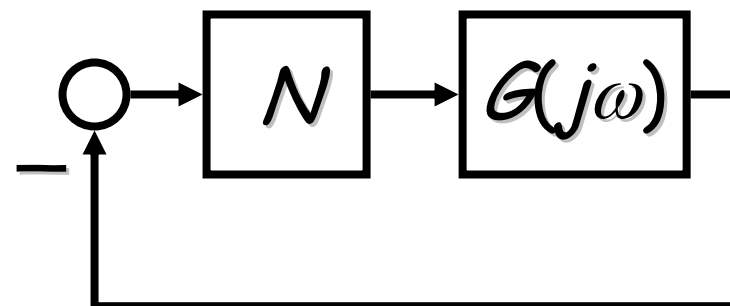
$$1 + G(j\omega)N(x_{\max}) = 0$$

$$G(j\omega)N(x_{\max}) = -1$$

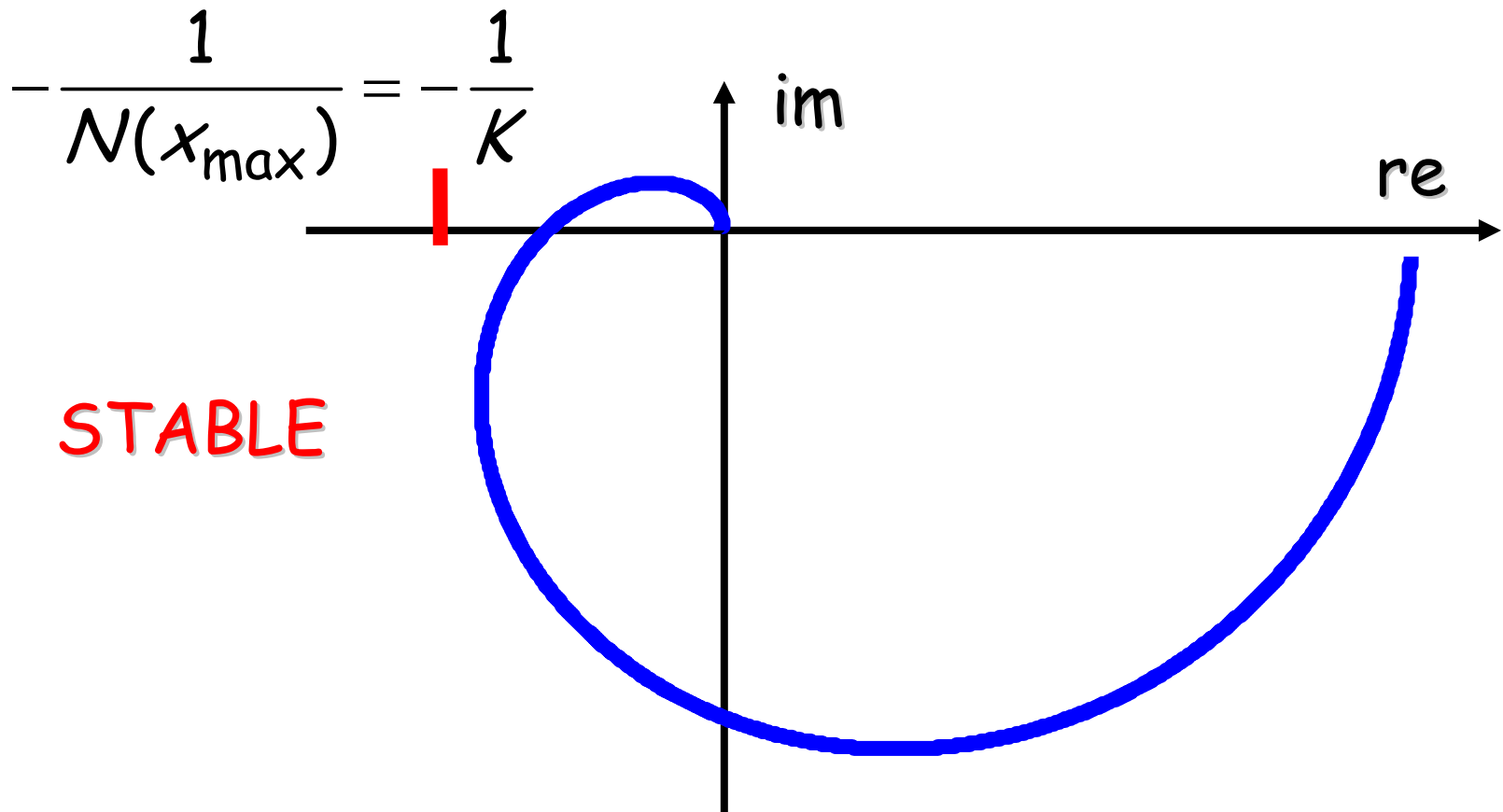
$$G(j\omega) = -\frac{1}{N(x_{\max})}$$

Nyquist plot

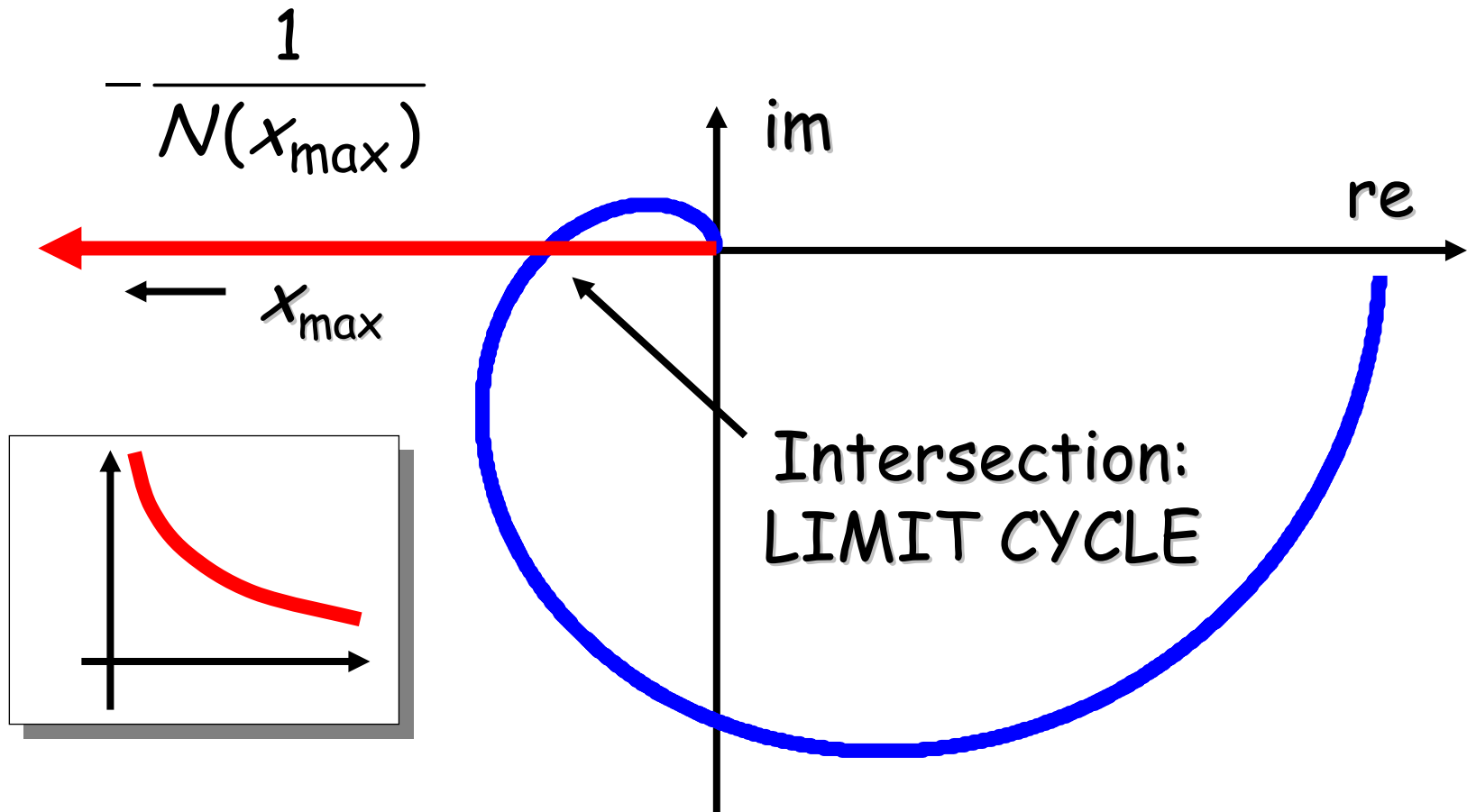
Inverse of DF

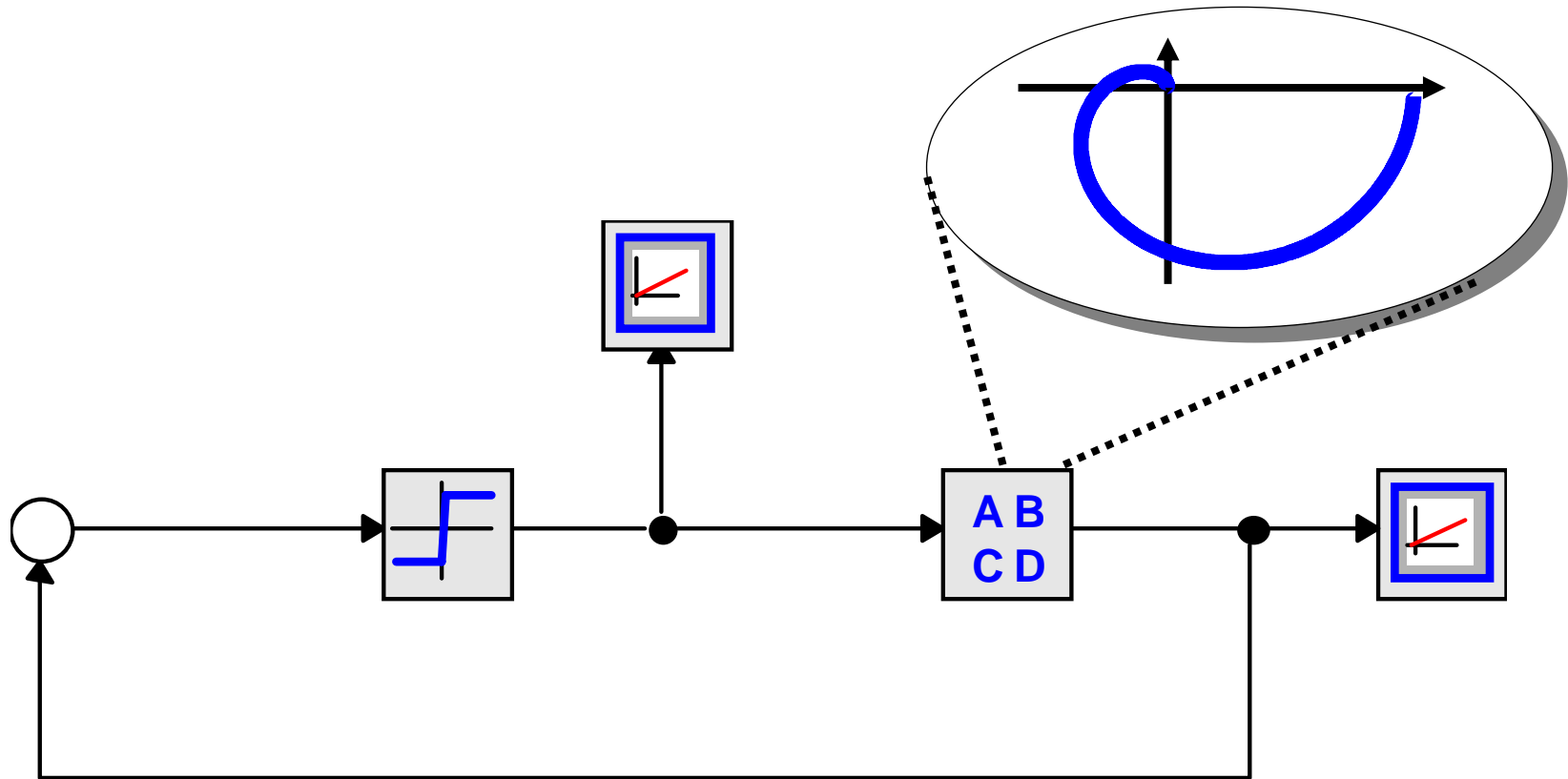


Example (gain)

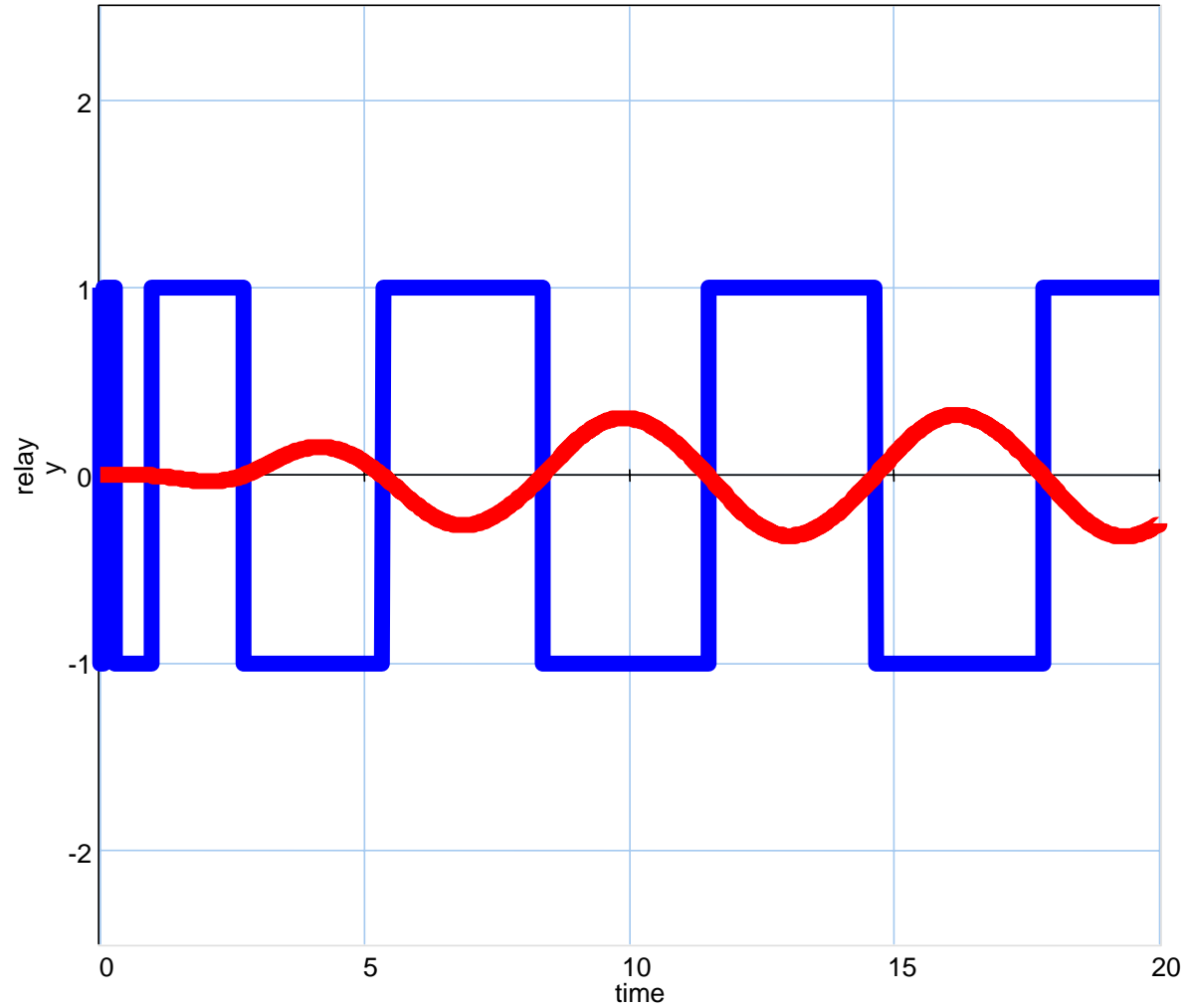


Example (relay)

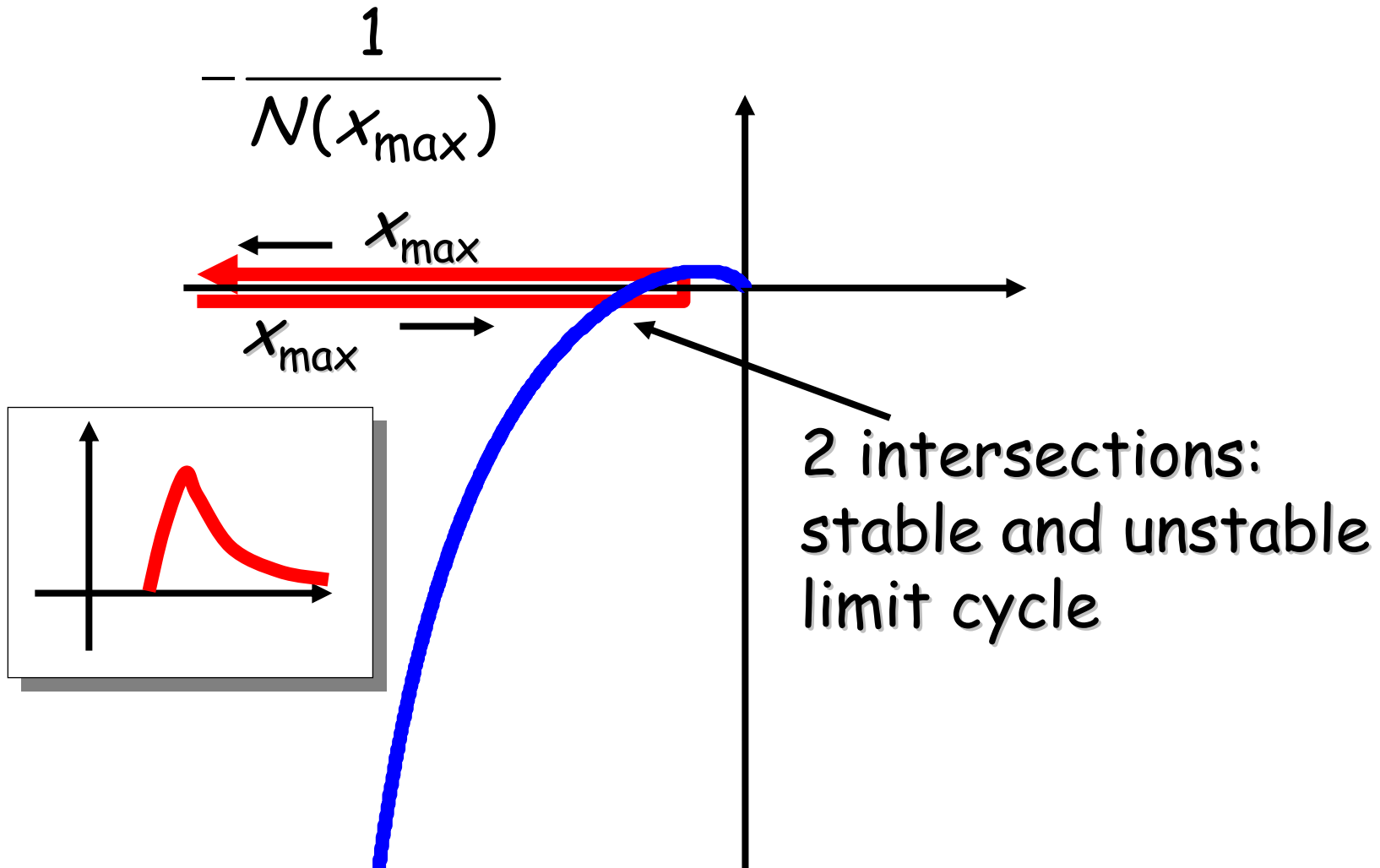




Limit_cycle_relay.em



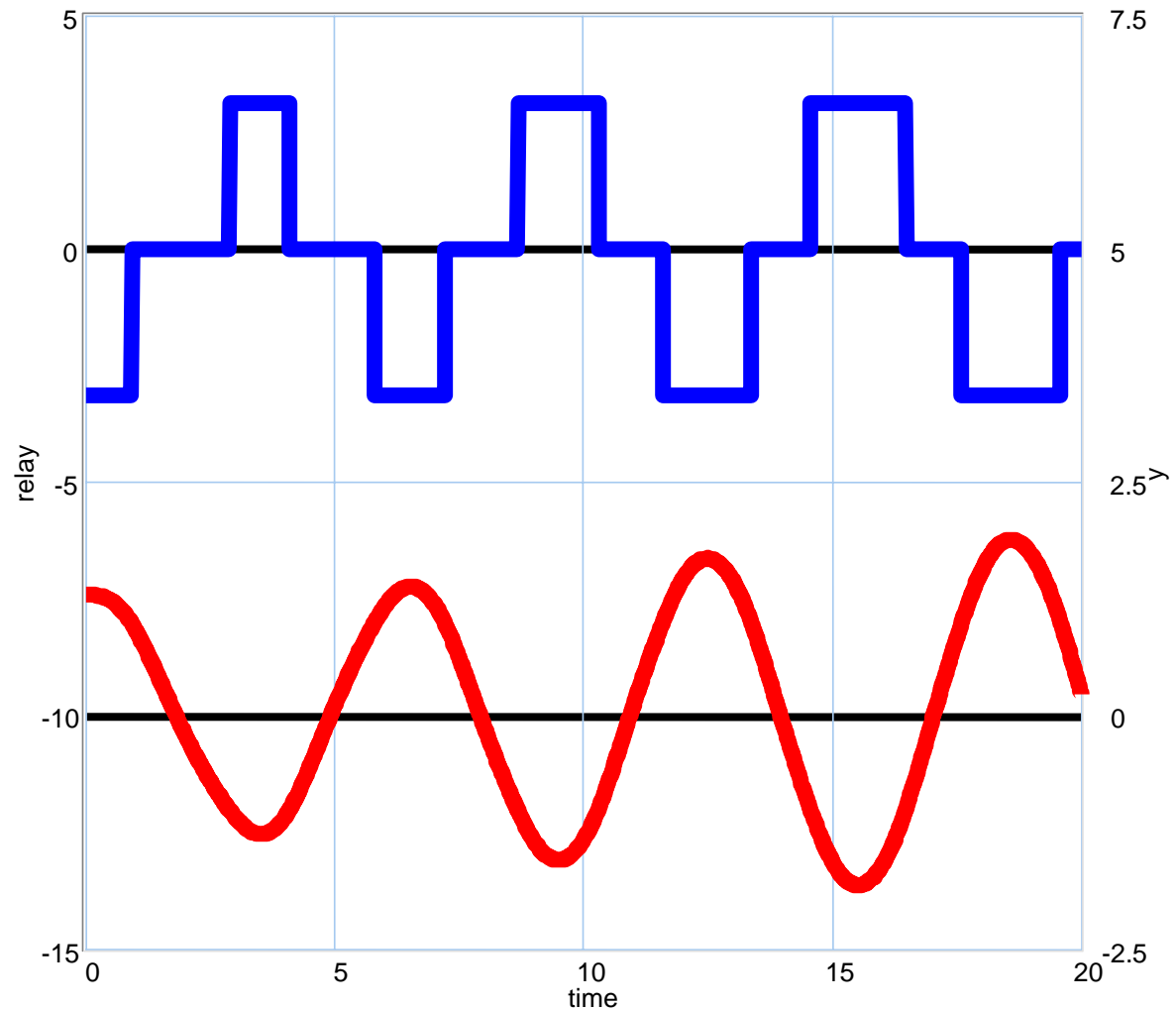
Example (relay + dead zone)



Response (1)

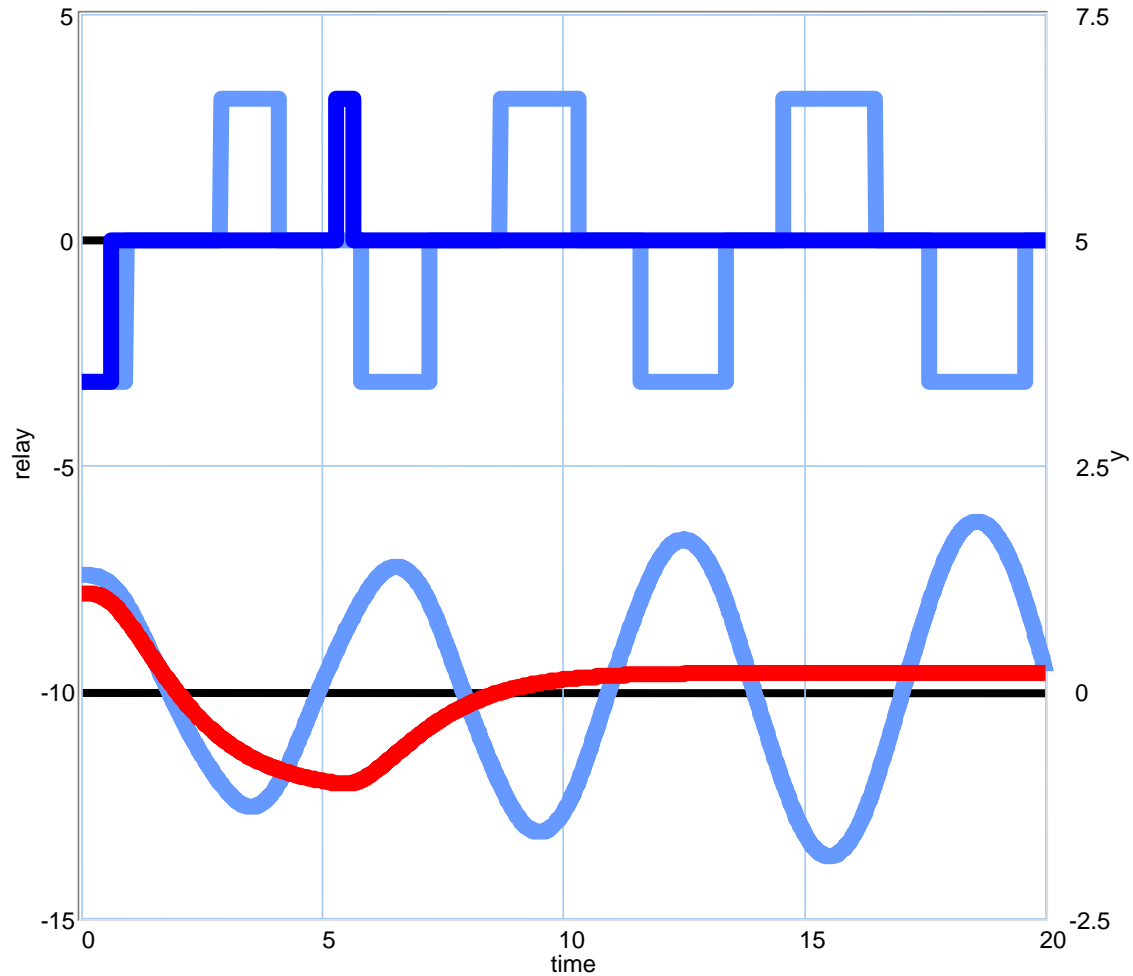
$y(0) < d (=1)$
no reaction

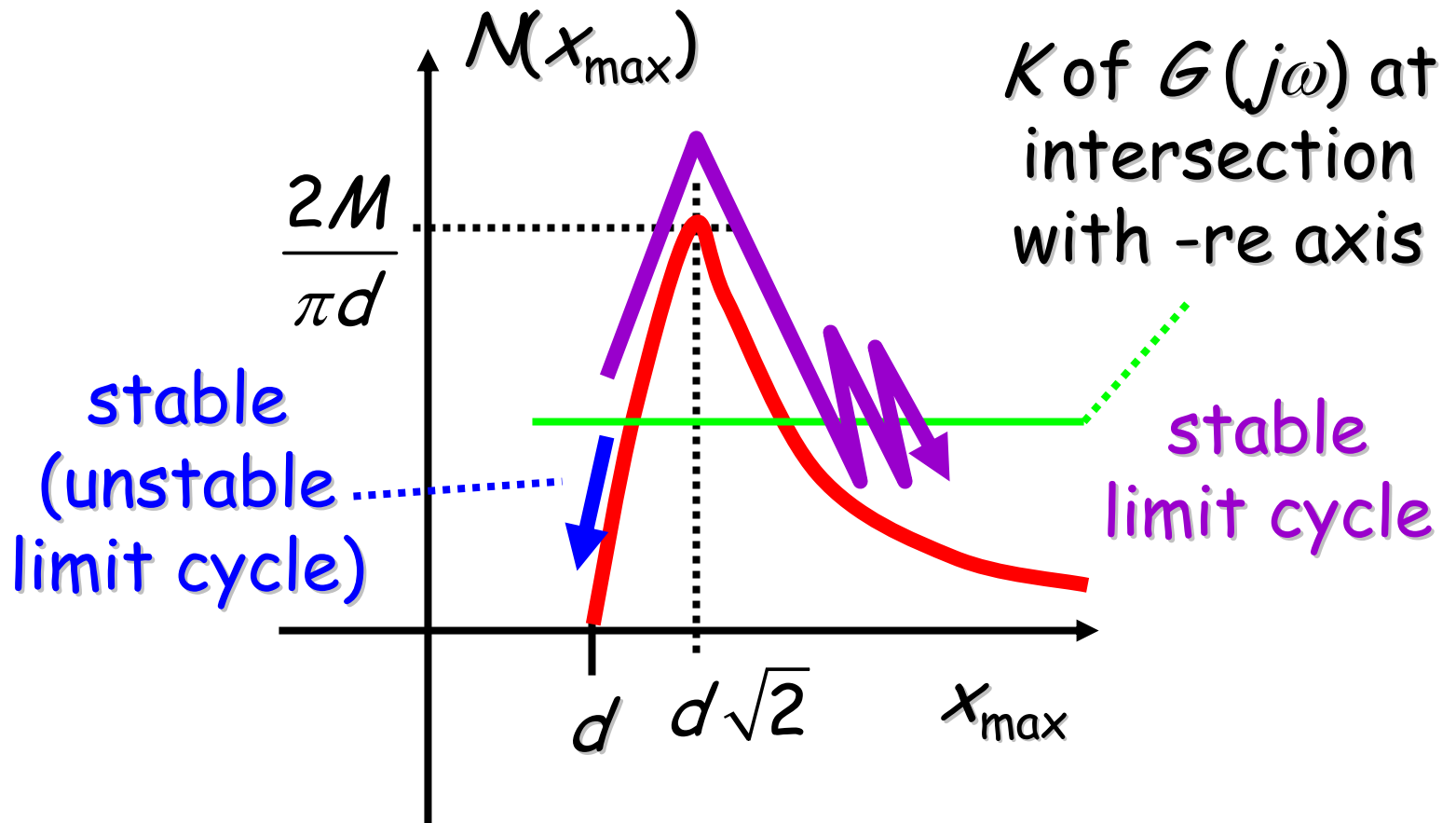
$y(0) = 1.3$



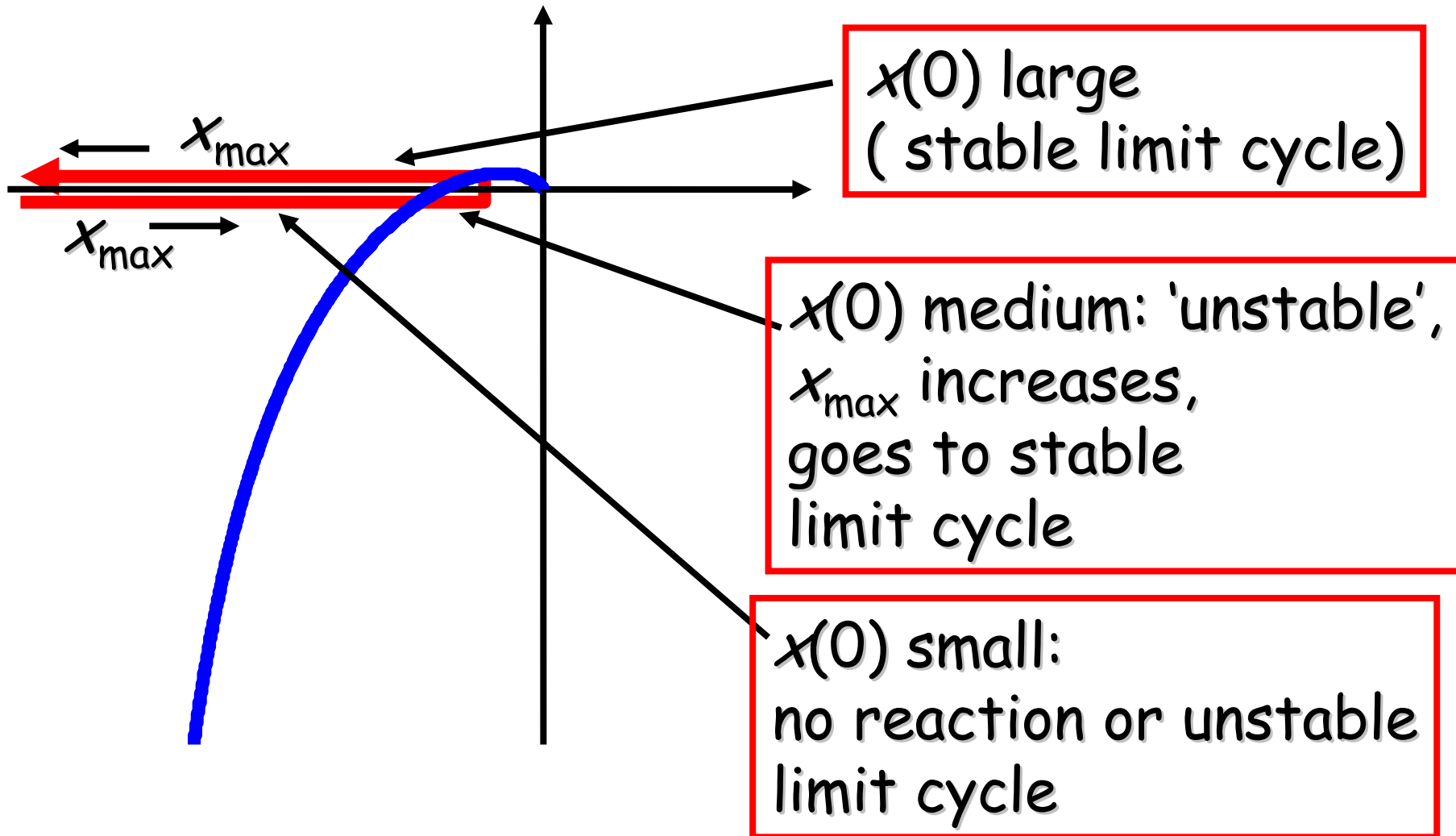
Response(2)

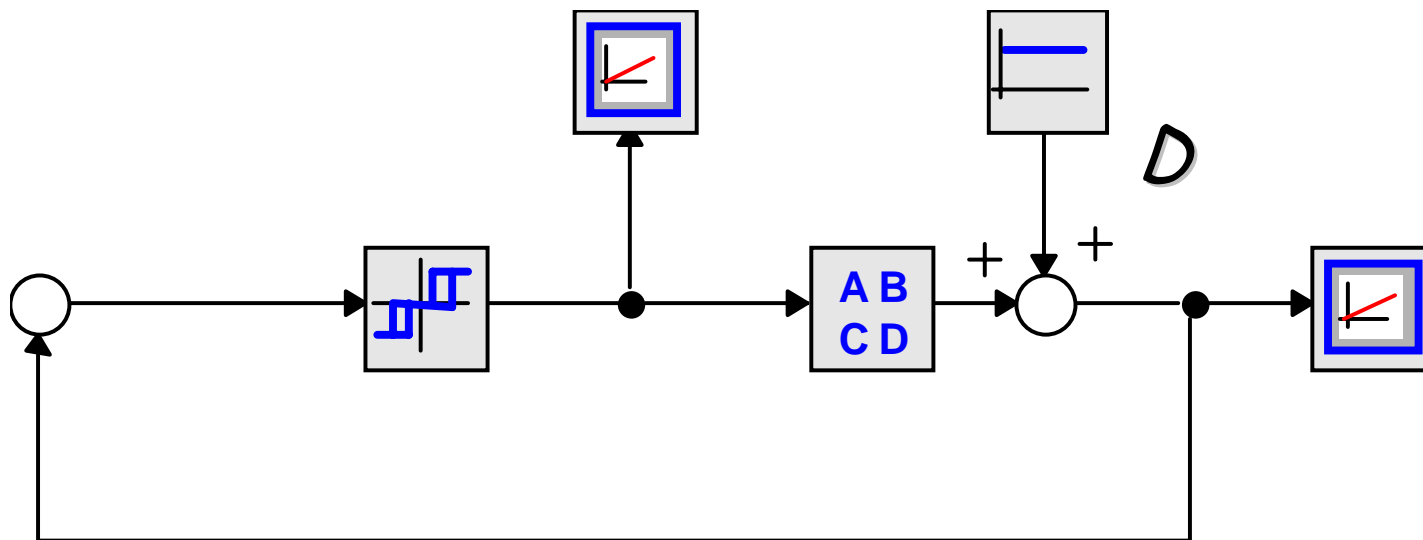
$$y(0) = 1.1$$





Example (relay + dead zone)





Limit_cycle_relay_dead_zone

Dead zone = 1
 $D = 0.8, 1.05, 1, 1.15$

- Describing function gives an interpretation of non-linear systems
 - only at border of instability !
 - Only valid for more or less sinusoidal signals
 - (only first harmonic of Fourier series is taken into account)