University of Twente
The Netherlands

Control Laboratory

Design in State Space (time domain)

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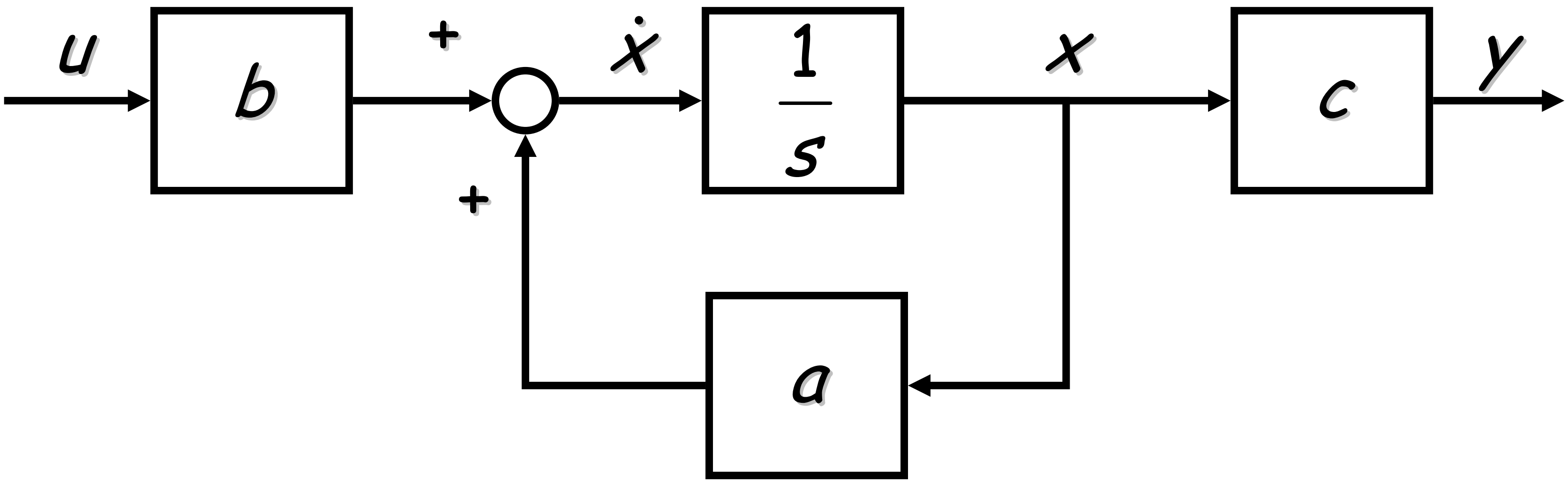
Contents

- State space description
- state feedback
- pole placement
- optimisation

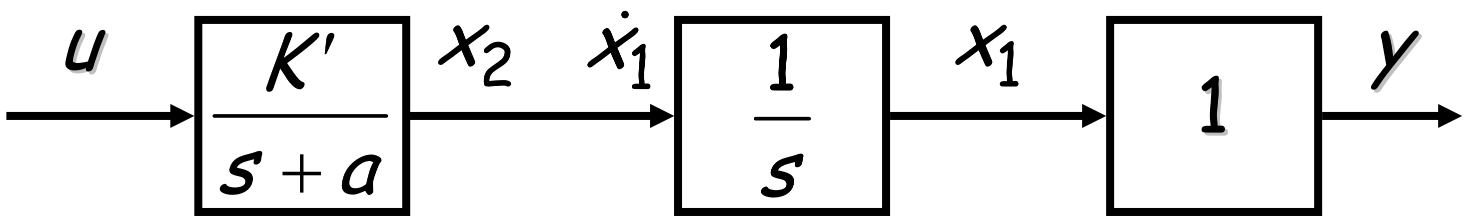
First-order system



$$\dot{x} = ax + bu$$

$$y = cx$$

Second order system



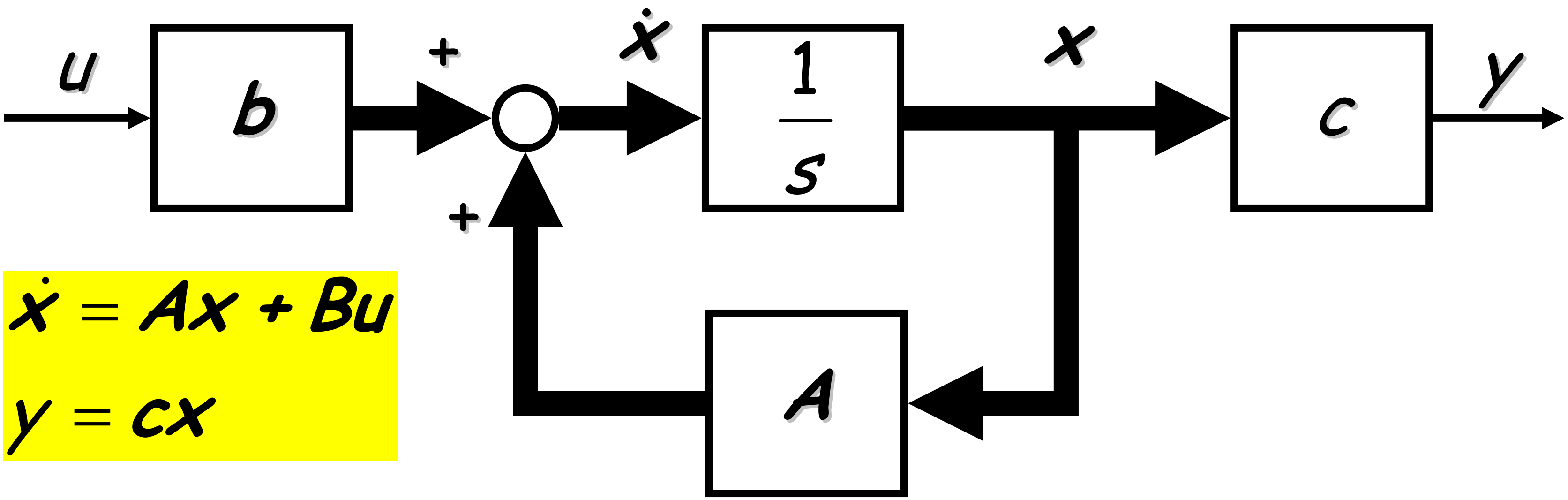
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -ax_2 + K'u$$
$$y = 1x_1$$

$$\longrightarrow \quad \dot{\boldsymbol{x}} = \begin{pmatrix} 0 & 1 \\ 0 & -a \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ K' \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \boldsymbol{x}$$

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u$$
$$y = \boldsymbol{c}\boldsymbol{x}$$

State-space description

Set of first-order systems

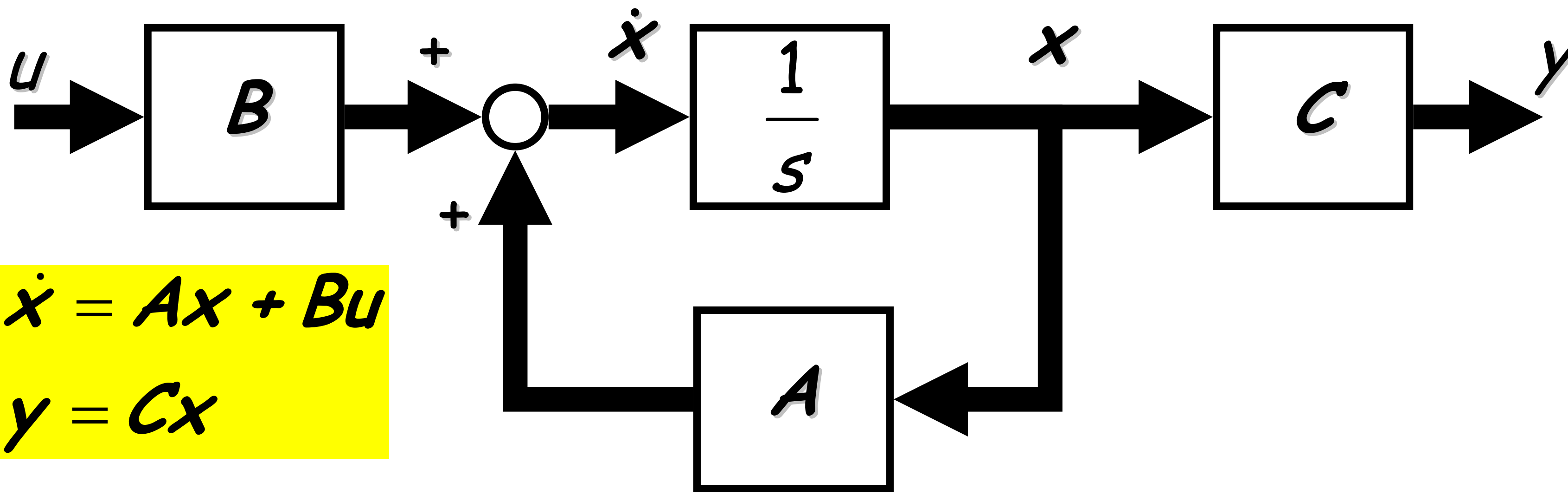


$$\dot{x} = Ax + Bu$$
$$y = cx$$

For SISO systems:
u = input signal (scalar)
y = output signal (scalar)
x = state vector ($n \times 1$)

For SISO systems:
A = system matrix ($n \times n$)
b = input matrix ($n \times 1$)
c = output matrix ($1 \times n$)

Set of first-order systems



$$\dot{x} = Ax + Bu$$
$$y = Cx$$

For MIMO systems:
u = input vector ($m \times 1$)
y = output signal ($p \times 1$)
x = state vector ($n \times 1$)

For MIMO systems:
A = system matrix ($n \times n$)
B = input matrix ($n \times m$)
C = output matrix ($p \times n$)

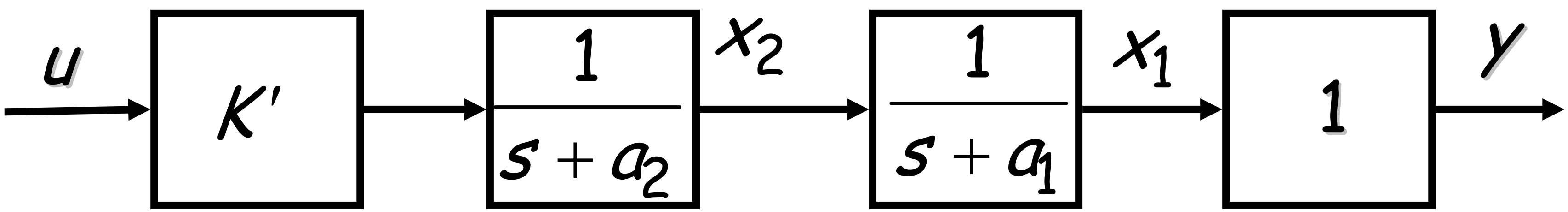
State (1)

- The state ($\boldsymbol{x}(t_0)$) of a system at $t = t_0$ is the minimal amount of information that is necessary to describe the behaviour of the system for $t > t_0$, if also the input(s) and the state equations are known

State (2)

- State variables are not unique
  - any linear combination of state variables is a state variable again
- E.g. the initial conditions of the integrators in the system

Series form



$$\dot{x}_1 = -a_1 x_1 + x_2$$
$$\dot{x}_2 = -a_2 x_2 + K'u$$
$$y = 1x_1$$

$$\longrightarrow \quad \dot{\boldsymbol{x}} = \begin{pmatrix} -a_1 & 1 \\ 0 & -a_2 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ K' \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \boldsymbol{x}$$

eigenvalues at the diagonal

Series form (alternative)

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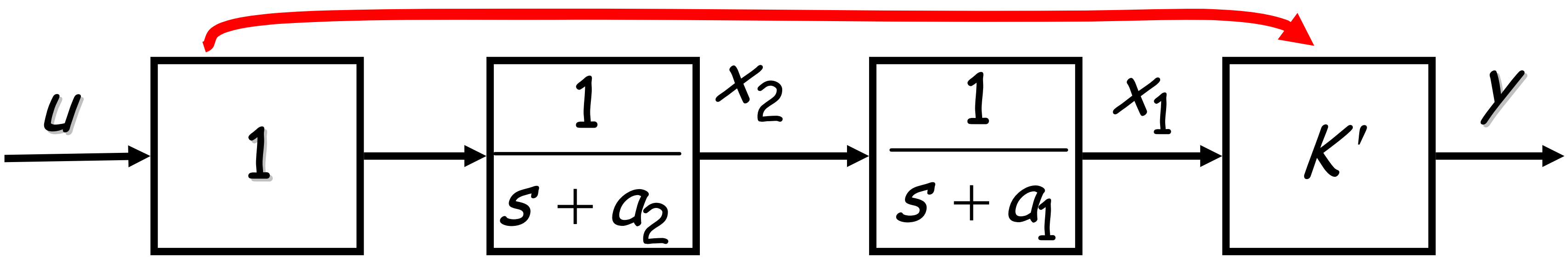
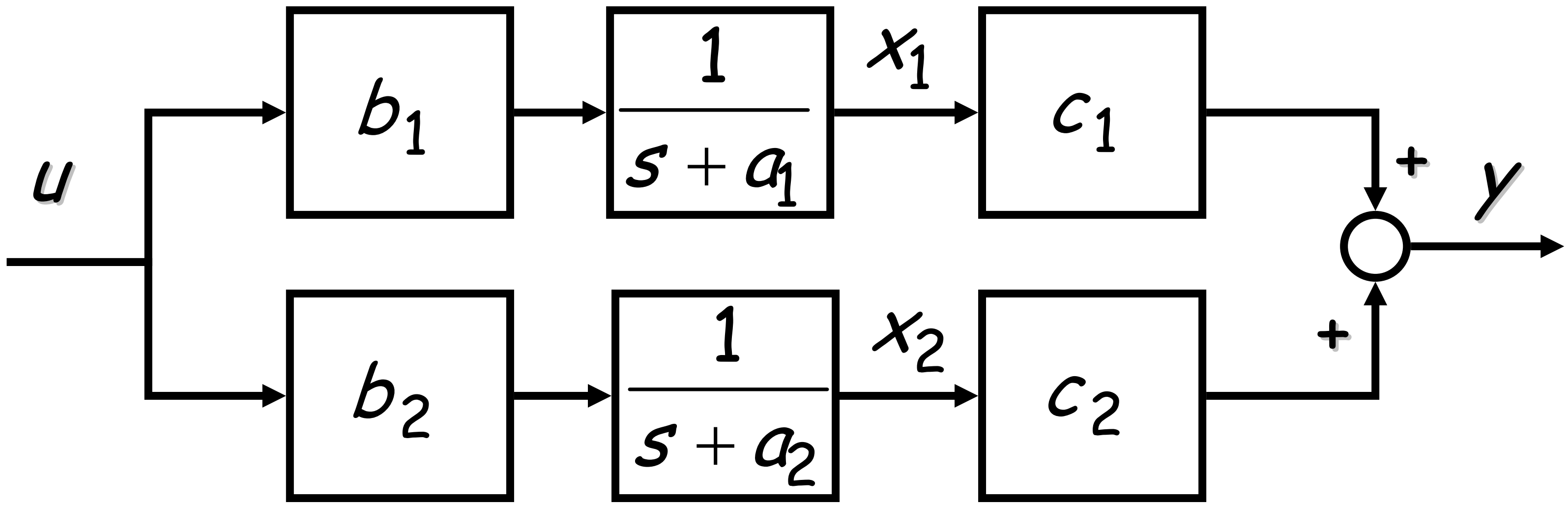


$$\dot{x}_1 = -a_1 x_1 + x_2$$
$$\dot{x}_2 = -a_2 x_2 + u$$
$$y = K' x_1$$

$$\longrightarrow \quad \dot{\boldsymbol{x}} = \begin{pmatrix} -a_1 & 1 \\ 0 & -a_2 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$
$$y = \begin{pmatrix} K' & 0 \end{pmatrix} \boldsymbol{x}$$

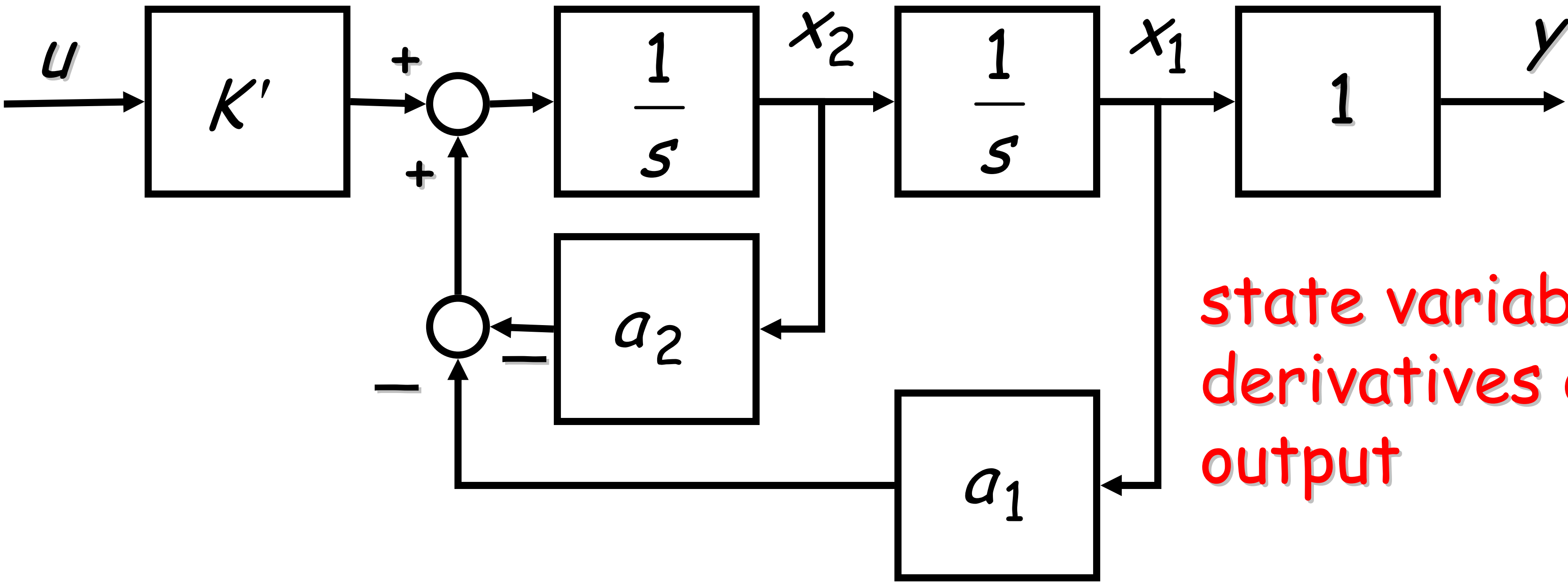


$$\dot{x}_1 = -a_1 x_1 + b_1 u$$
$$\dot{x}_2 = -a_2 x_2 + b_2 u$$
$$y = c_1 x_1 + c_2 x_2$$

$$\longrightarrow \quad \dot{\boldsymbol{x}} = \begin{pmatrix} -a_1 & 0 \\ 0 & -a_2 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} u$$
$$y = \begin{pmatrix} c_1 & c_2 \end{pmatrix} \boldsymbol{x}$$

eigenvalues at the diagonal

Phase-variable form



state variables derivatives of output

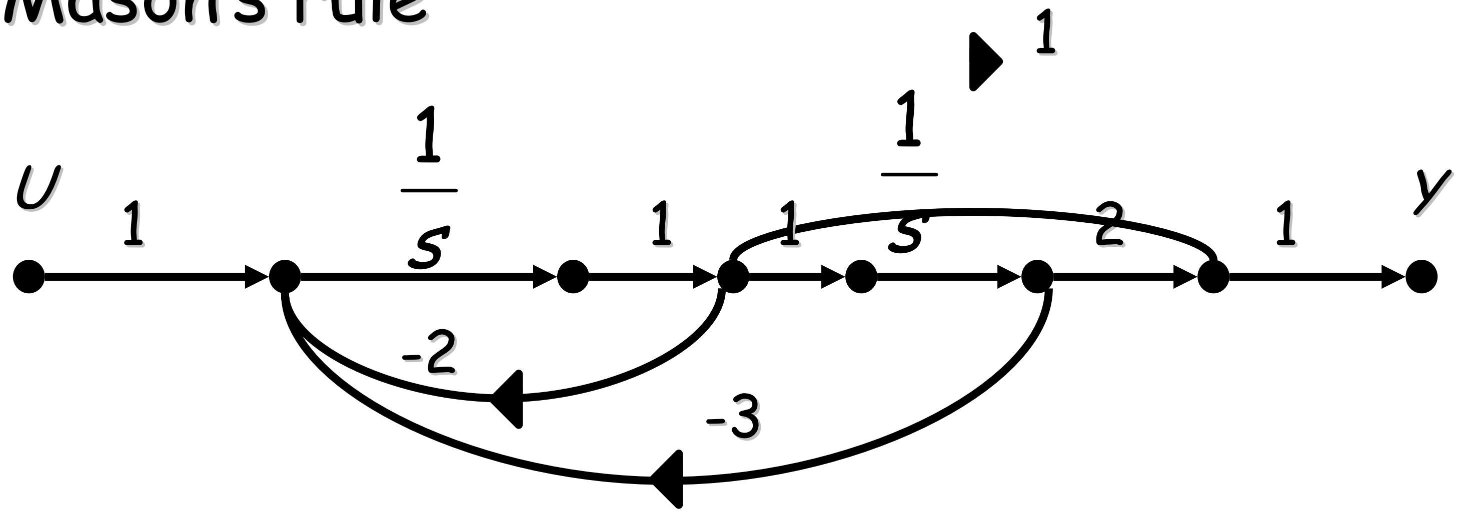
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a_1 x_1 - a_2 x_2 + K' u$$
$$y = x_1$$

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 0 & 1 \\ -a_1 & -a_2 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ K' \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \boldsymbol{x}$$

Phase-variable form (zeros)

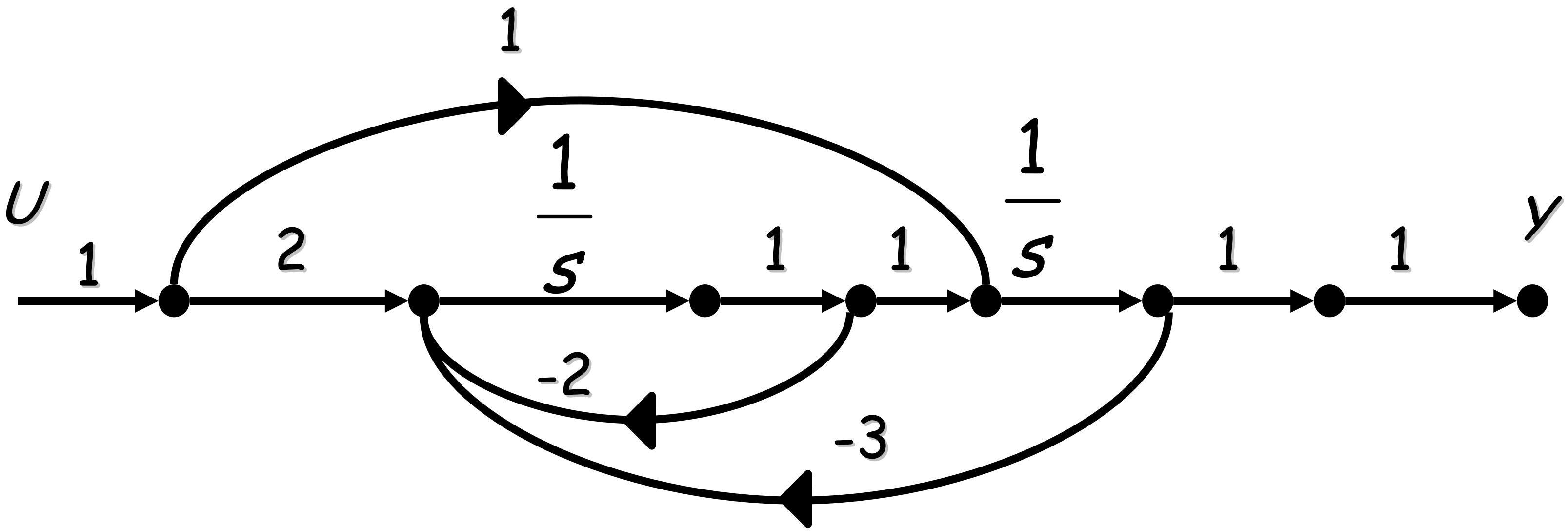
$$\frac{s+2}{s^2+2s+3} = \frac{\dfrac{1}{s}+\dfrac{2}{s^2}}{1+\dfrac{2}{s}+\dfrac{3}{s^2}} = \frac{P_1+P_2}{1-L_1-L_2}$$

Mason's rule



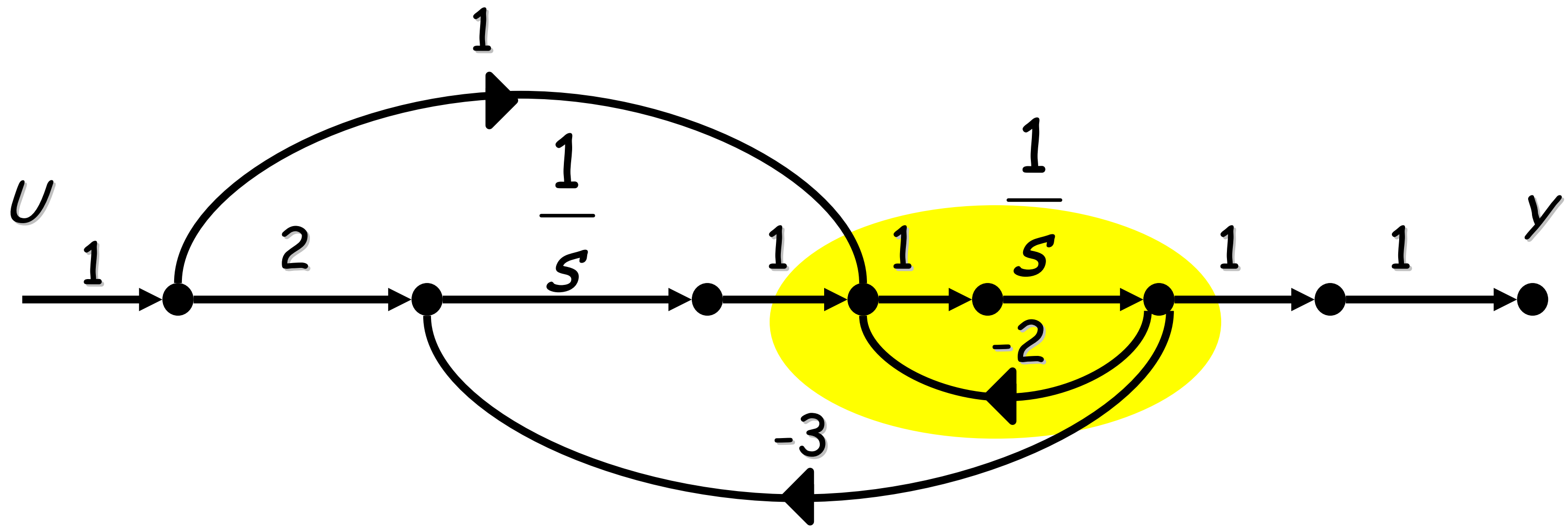
Phase-variable form (alternative)

$$\frac{s+2}{s^2+2s+3} = \frac{\dfrac{1}{s}+\dfrac{2}{s^2}}{1+\dfrac{2}{s}+\dfrac{3}{s^2}} = \frac{P_1+P_2}{1-L_1-L_2}$$

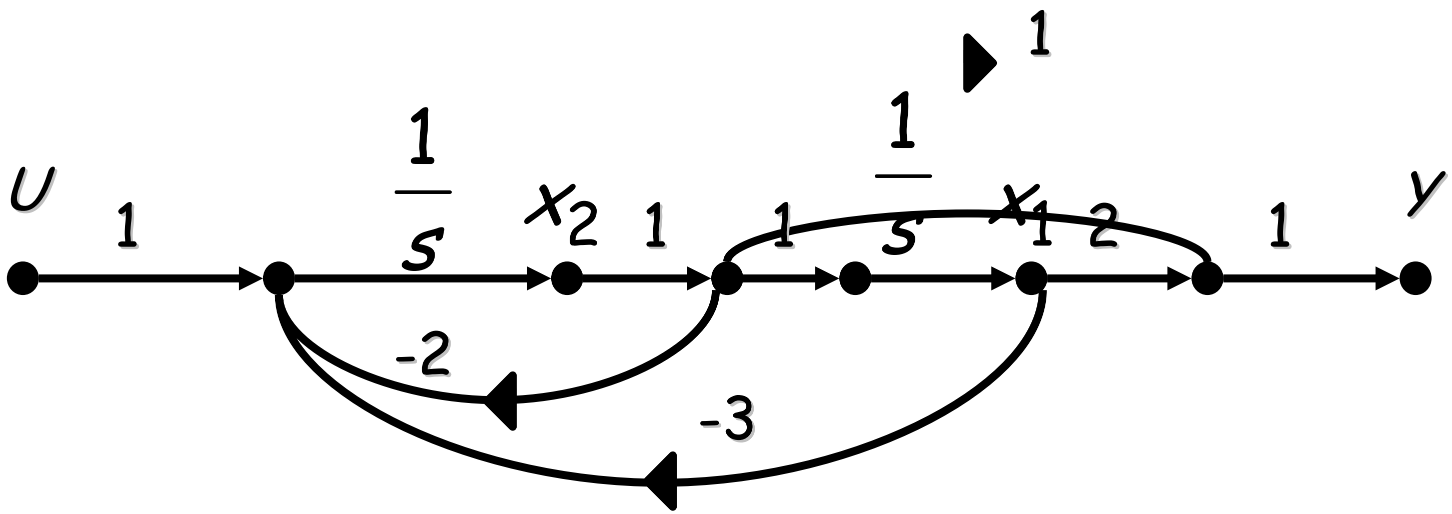


Dual phase-variable form

$$\frac{s+2}{s^2+2s+3} = \frac{\frac{1}{s}+\frac{2}{s^2}}{1+\frac{2}{s}+\frac{3}{s^2}} = \frac{P_1+P_2}{1-L_1-L_2}$$



Phase variable form



$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ & & & & 1 \\ -a_1 & -a_2 & \cdots & \cdots & -a_n \end{pmatrix}$$

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 2 & 1 \end{pmatrix} x$$

zero

Phase-variable form (alternative)

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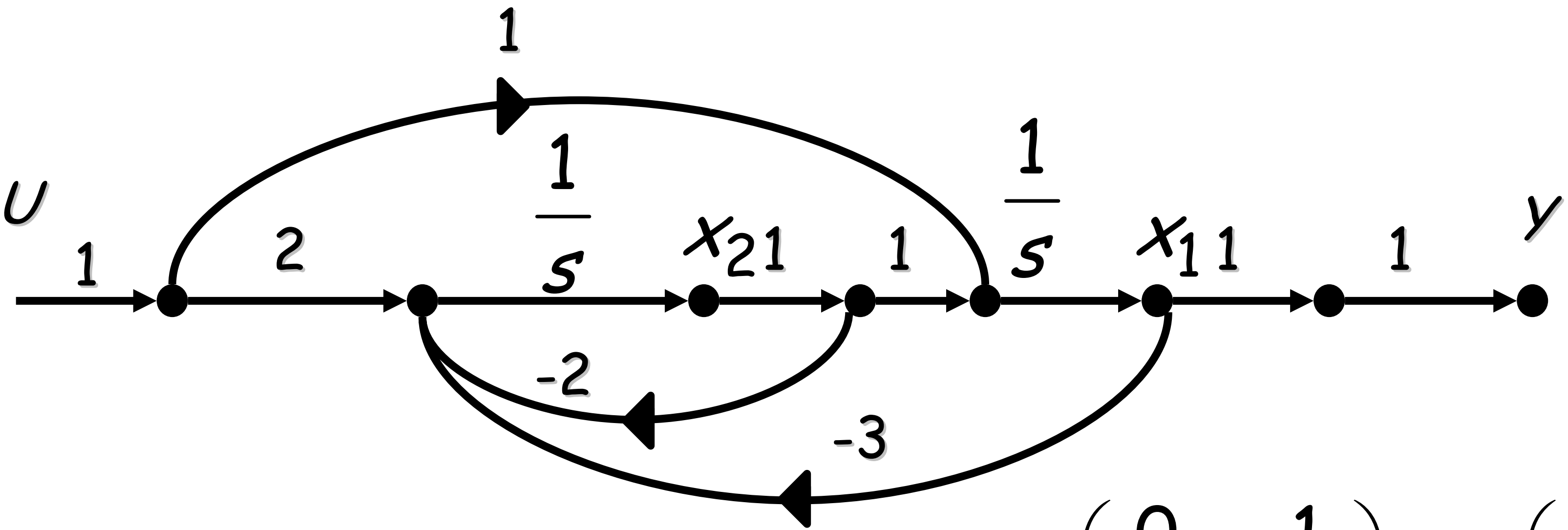

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

zero

Dual phase-variable form

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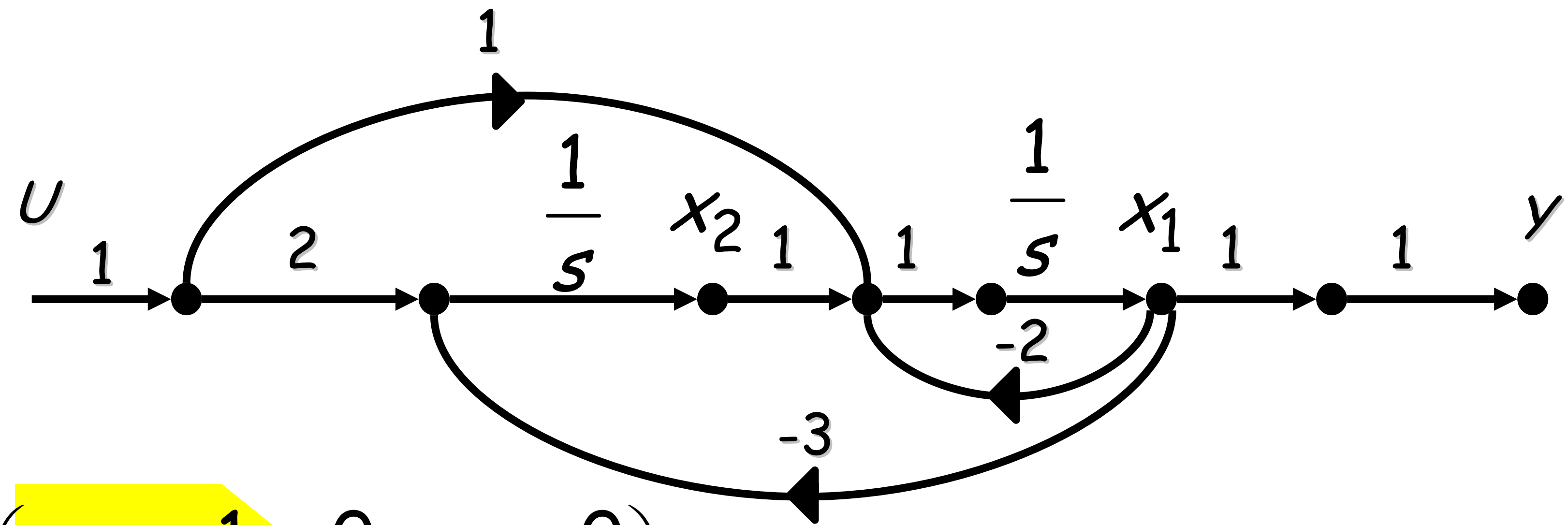


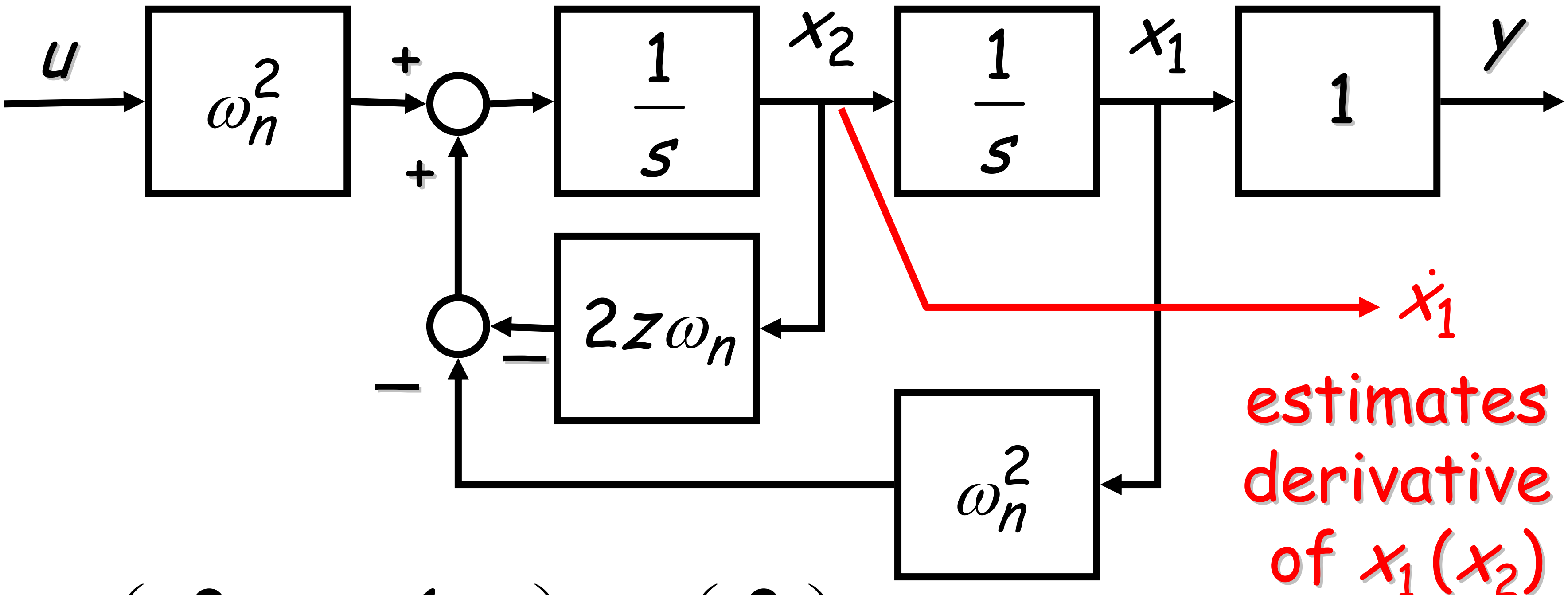
$$A = \begin{pmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ & & & & 1 \\ -a_n & & \cdots & \cdots & 0 \end{pmatrix}$$

$$\dot{x} = \begin{pmatrix} -2 & 1 \\ -3 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

zero

State-variable filter



$$\dot{\boldsymbol{x}} = \begin{pmatrix} 0 & 1 \\ -\omega_n^2 & -2z\omega_n \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ \omega_n^2 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \boldsymbol{x}$$

if $\quad y = \begin{pmatrix} 0 & 1 \end{pmatrix} \boldsymbol{x}$

$$y = \dot{x}_1 \approx \dot{u}$$

SVF demos

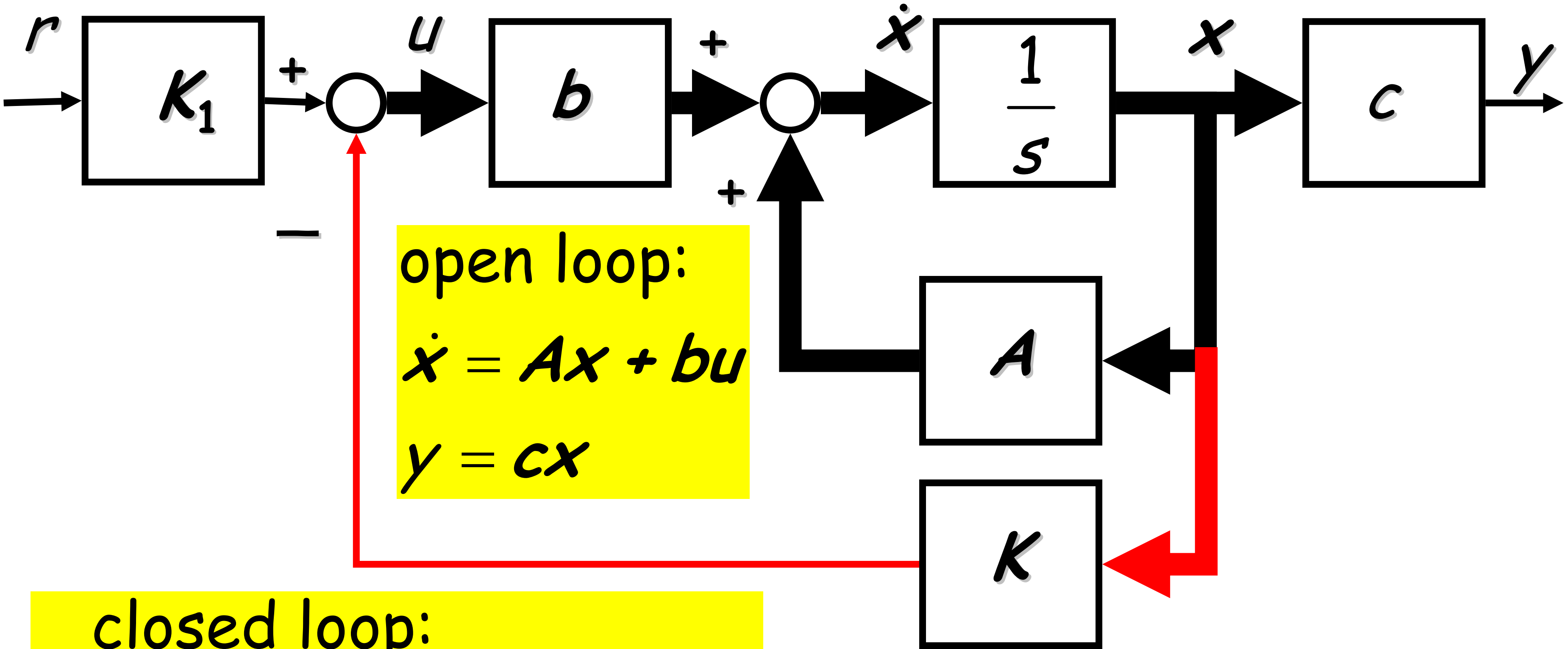
Demo SVF bandwidth
Demo_SVF

20-sim

Demo SVF noise
Demo_SVF_noise

State space design

State Feedback



open loop:

$$\dot{x} = Ax + bu$$
$$y = cx$$

closed loop:

$$\dot{x} = \left(A - bK\right)x + bK_1 r$$
$$y = cx$$

Pole placement

$$\dot{\boldsymbol{x}} = \left(\boldsymbol{A} - b\boldsymbol{K}\right)\boldsymbol{x} + K_1 br$$

$$A' = \left(\boldsymbol{A} - b\boldsymbol{K}\right)$$

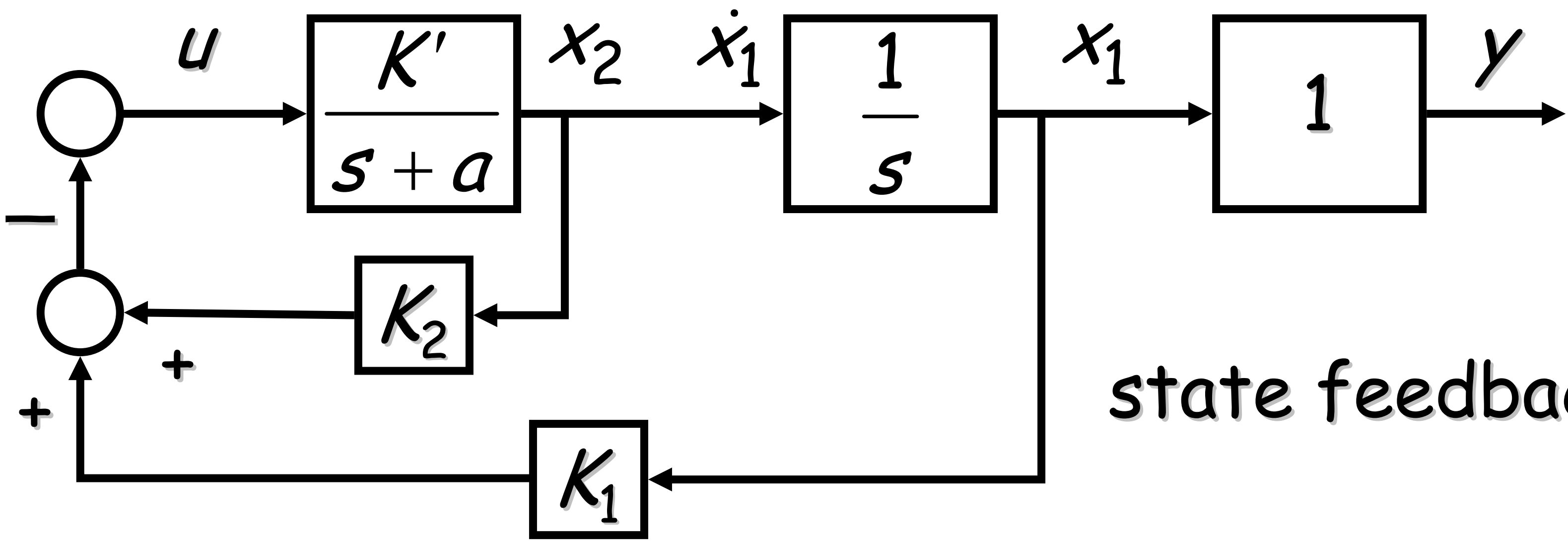
When K is properly chosen,
A' can get any desired eigen values

Poles can be placed by means of state feedback

(stable) zeros can only be relocated by means of prefilter

Example

Consider the process: $\dot{\boldsymbol{x}} = \begin{pmatrix} 0 & 1 \\ 0 & -a \end{pmatrix}\boldsymbol{x} + \begin{pmatrix} 0 \\ K' \end{pmatrix}u$

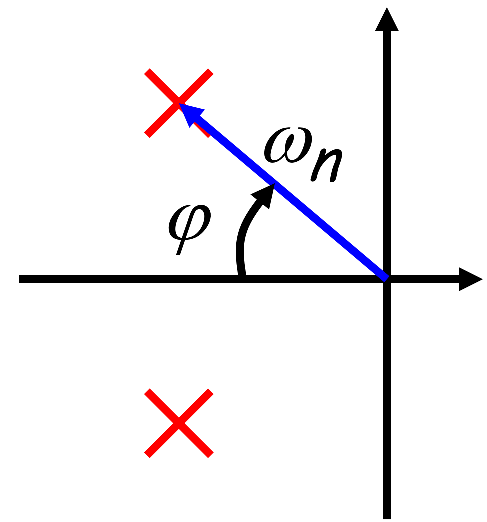


state feedback

$$A' = \begin{pmatrix} 0 & 1 \\ -K'K_1 & -a - K'K_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_n^2 & -2z\omega_n \end{pmatrix}$$

Example

$$A' = \begin{pmatrix} 0 & 1 \\ -K'K_1 & -a-K'K_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_n^2 & -2z\omega_n \end{pmatrix}$$



$$\boxed{K'K_1 = \omega_n^2} \qquad \boxed{K'K_2 = 2z\omega_n - a}$$

$$a + K'K_2 = 2z\omega_n \qquad = 2z\sqrt{K'K_1} - a$$

$$z = \cos(\varphi)$$

$$\text{if } K' = 1,\ a = 1 \qquad \omega_n = 2,\ z = 0.7$$

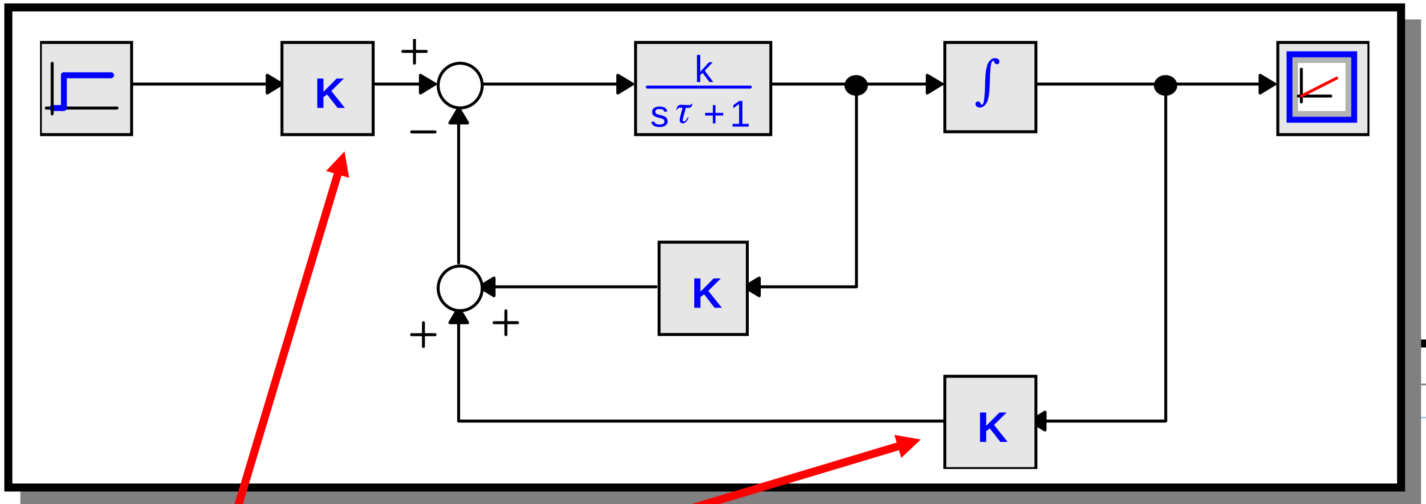
process

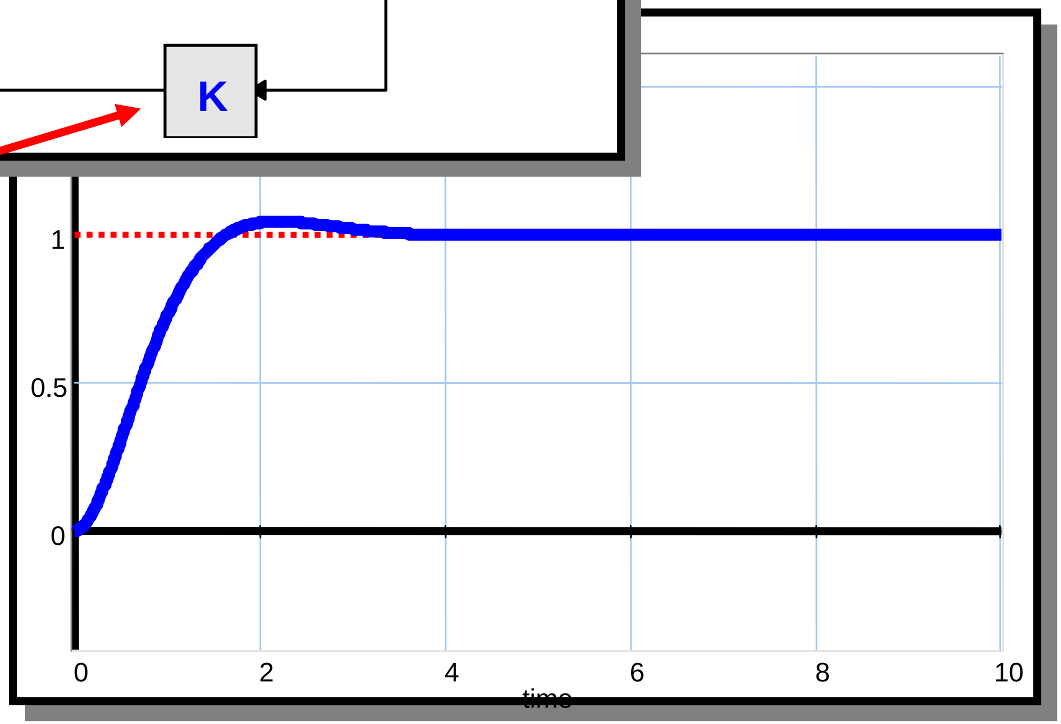
Design choice

$$K_1 = 2^2 = 4$$

$$K_2 = 2z2 - 1 = 1.8$$

Simulation





Should be equal to guarantee that $C/R \rightarrow 1$

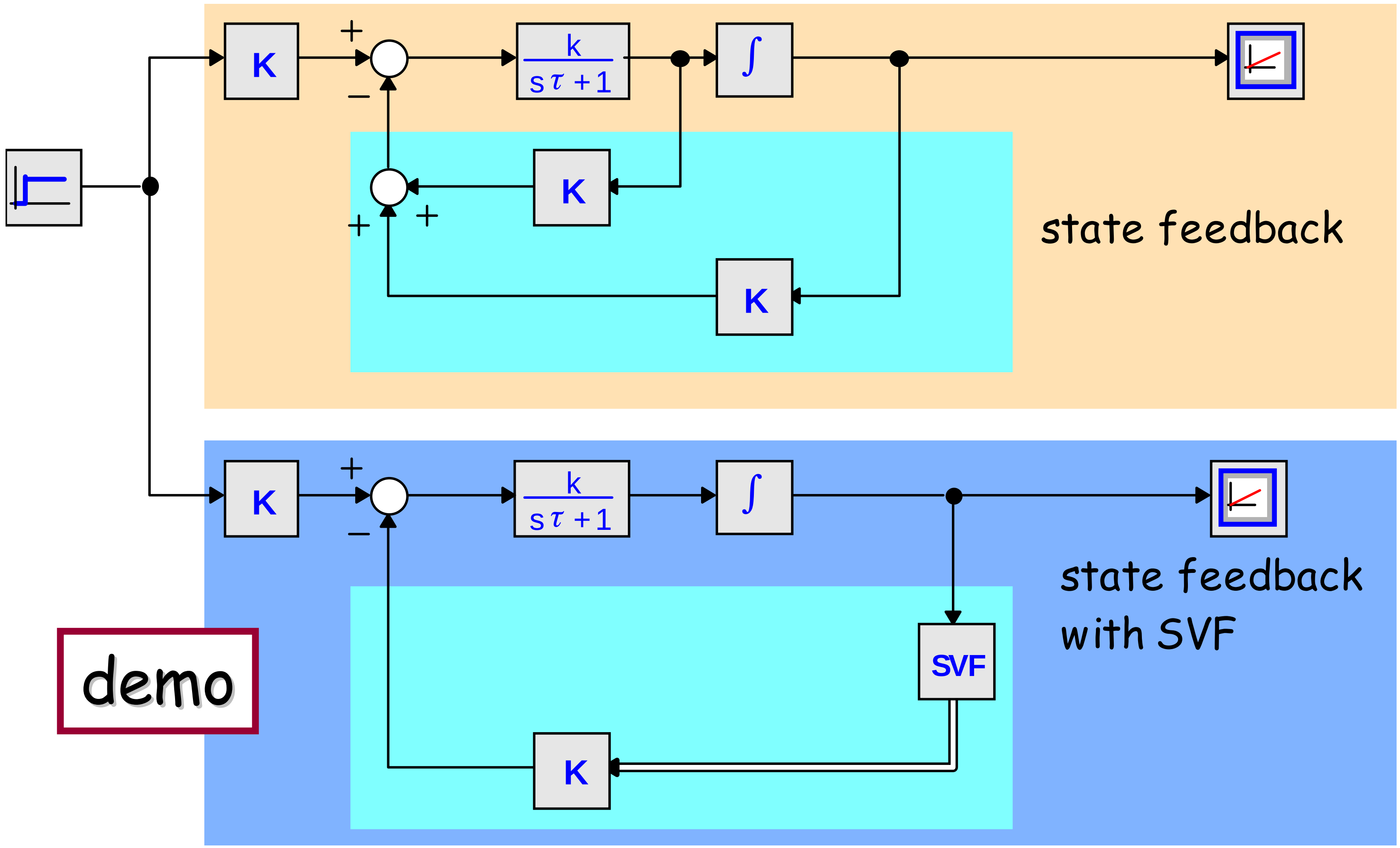
All states must be available

- State feedback assumes that all states can be used for feedback…
- This implies that

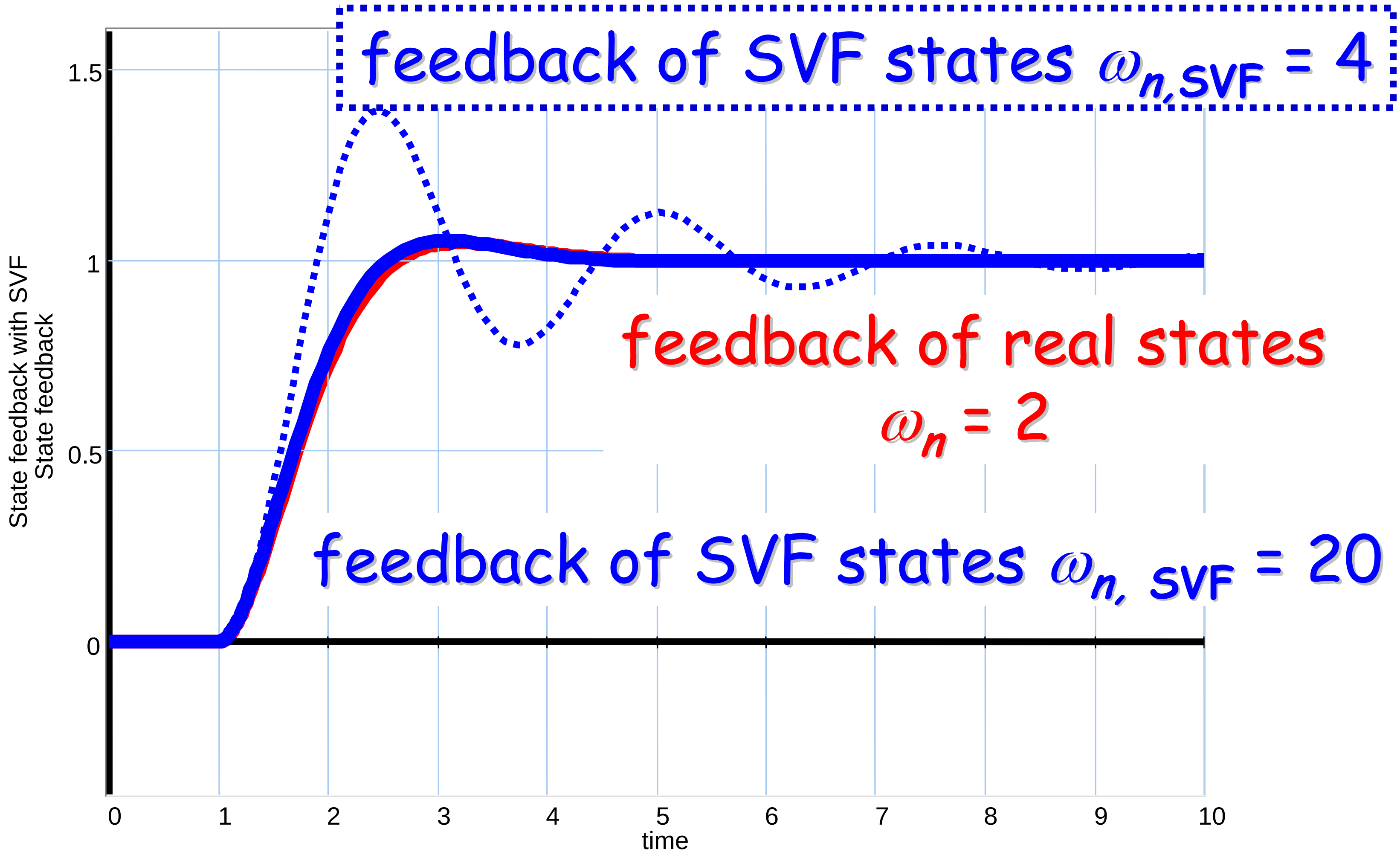
$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{x}$$

- If all states are not available they can be estimated
  - e.g. with a state variable filter (SVF)

Demonstration 20-sim



Responses


feedback of SVF states $\omega_{n,SVF} = 4$
feedback of real states
$\omega_n = 2$
feedback of SVF states $\omega_{n,\ SVF} = 20$
State feedback with SVF
State feedback
1.5
1
0.5
0
0
1
2
3
4
5
6
7
8
9
10
time

Rule of thumb

- If the bandwidth of the SVF is chosen 10 times larger than the bandwidth of the controlled process, the phase lag of the SVF is negligible.
- Can only be done when there is (almost) no noise on measured *y*
- Course ‘Digital Control’ will give more advanced solutions

System performance

- Performance of a system can be expressed in terms of
  - bandwidth
  - pole locations (in fact the same)
  - optimal control problem

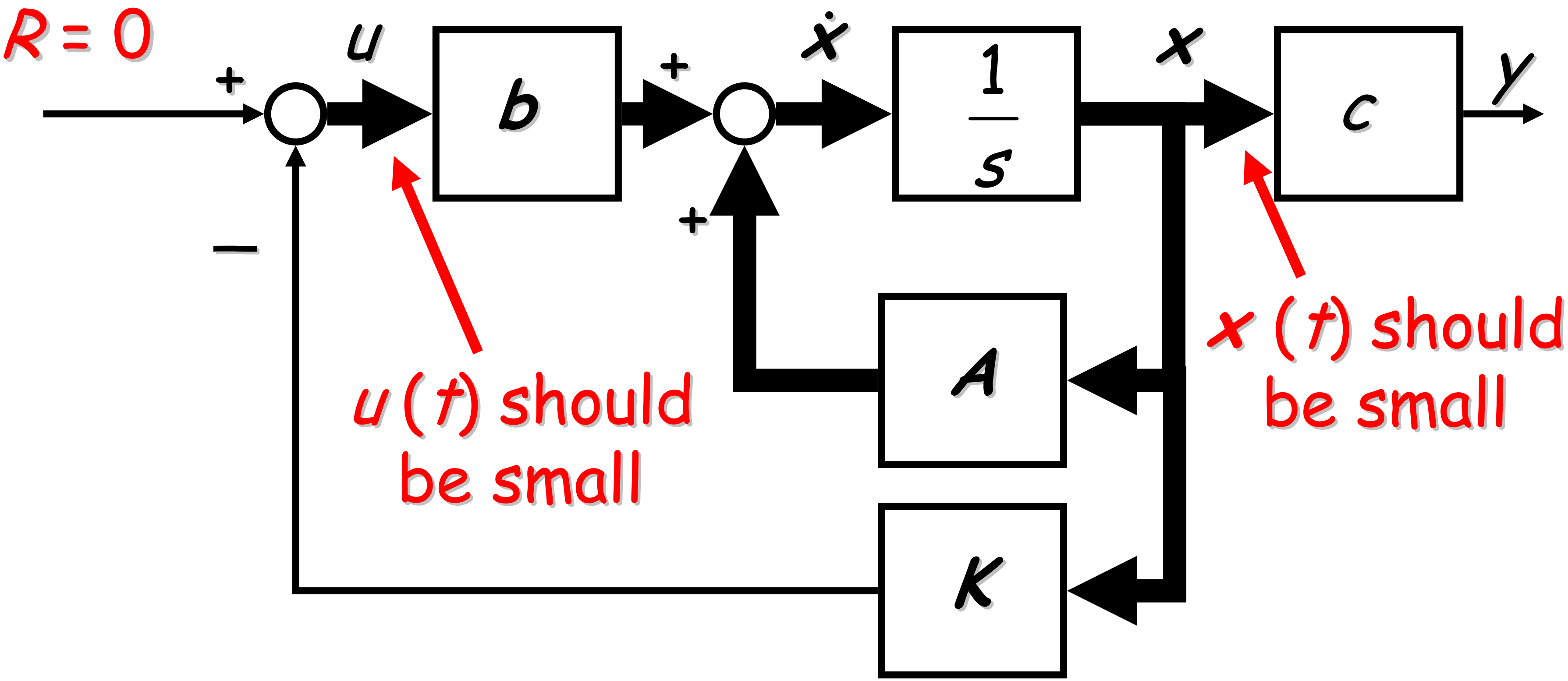
Trade off

- Error should be small
- reference changes should be perfectly tracked

But

- not at any price:
- control effort should be kept small
  - energy
  - price of equipment

Regulator system



Consider errors at $t = 0$ ($x(0) \neq 0$)

Optimisation

We consider the system

$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + bu$, with state feedback

$u = -\boldsymbol{K}\boldsymbol{x}$

System description

Find the feedback gain, K, such that

Adjustable parameter(s)

$$J = \int_0^\infty \left(\boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} + r u^2 \right) dt \text{ is minimal}$$

Criterion

Quadratic criterion

Second order system

If we consider a second-order system and

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

then $$J = \int_0^\infty \left(x^2 + ru^2 \right) dt$$

Meaningful criteria

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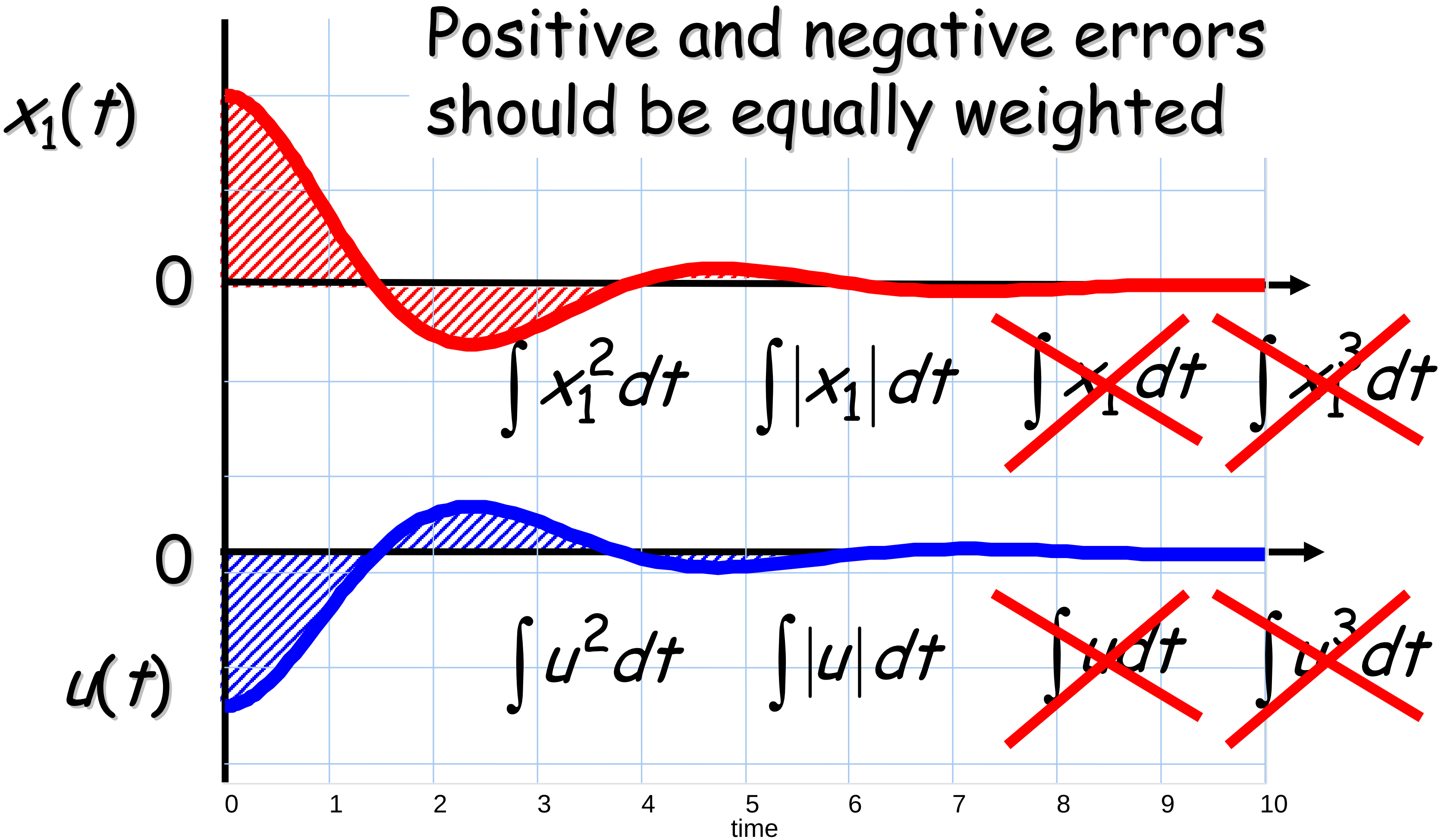
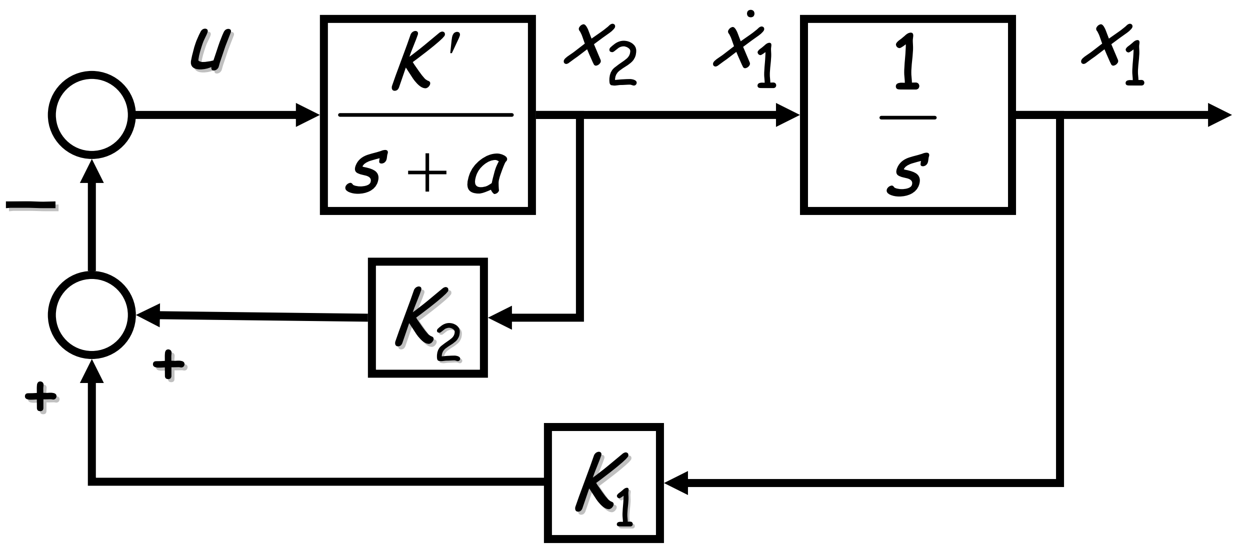


Not properly defined optimisation problem

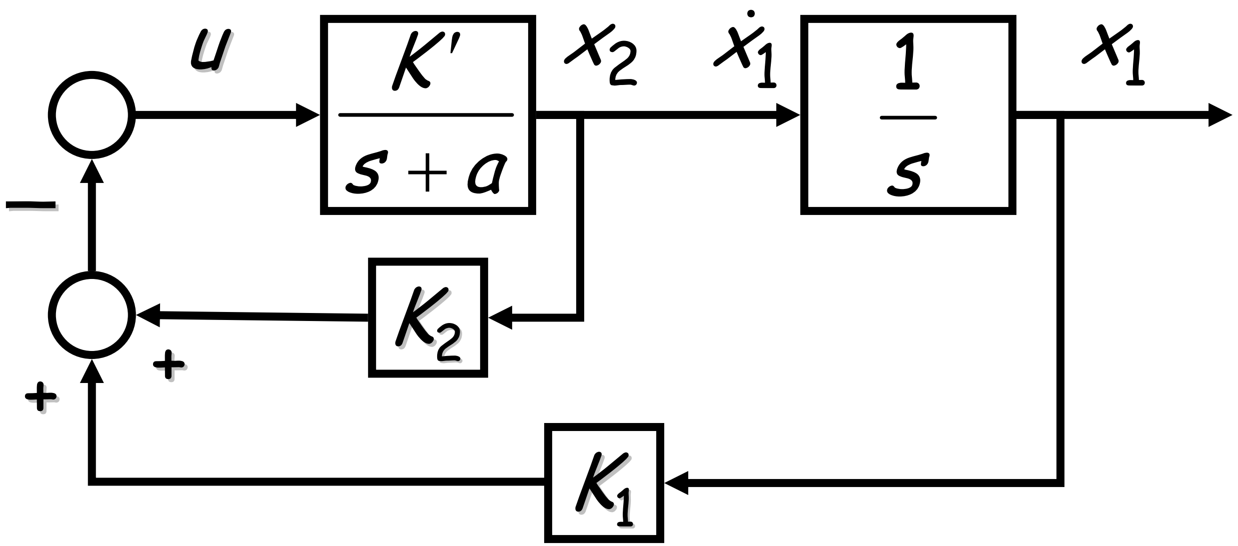


Find the feedback gains, K_1, K_2, such that

$\int x_1^2 dt$ is minimal

K_1, K_2 go to ∞

Properly defined optimisation problem



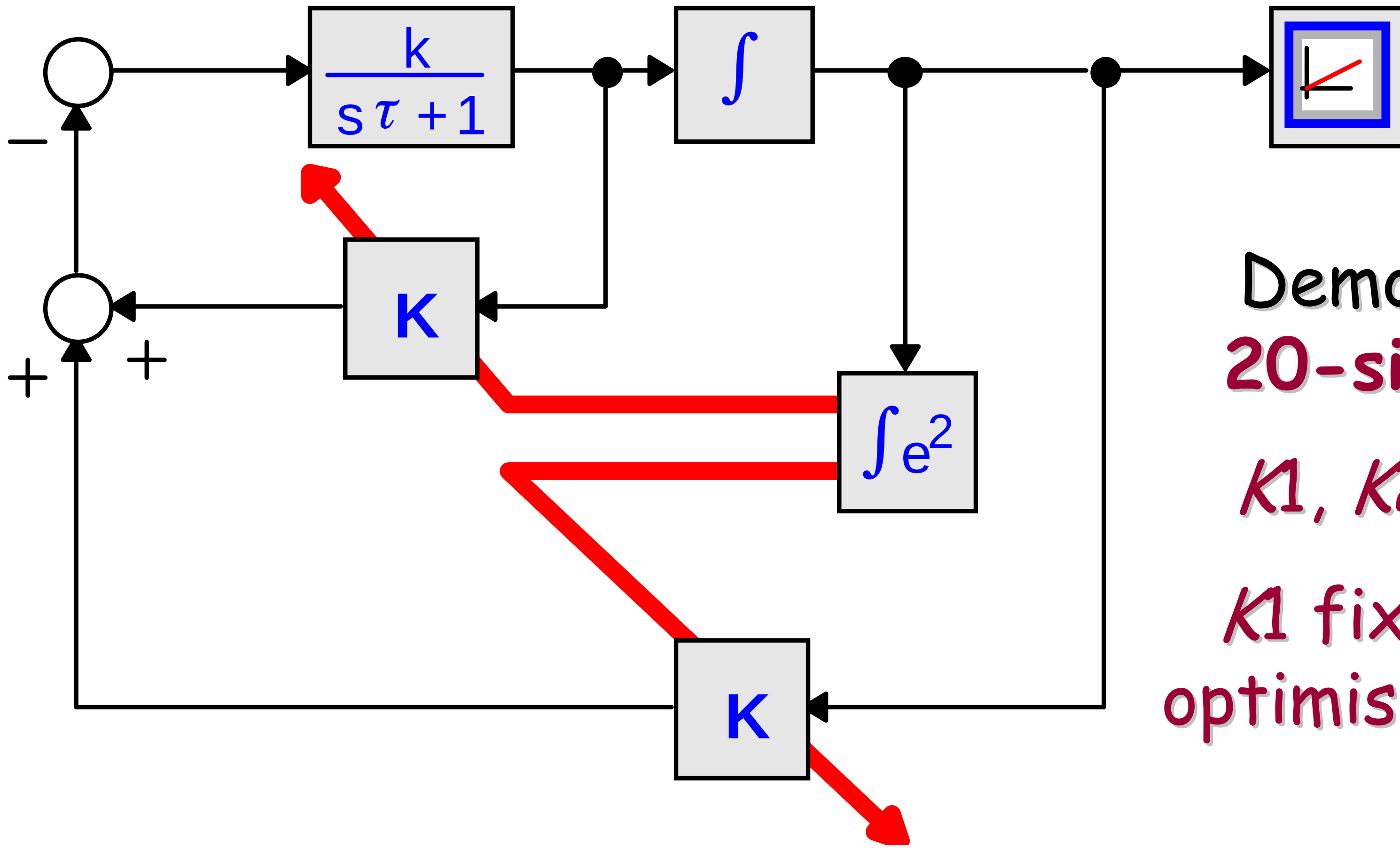
Given K_1, find the feedback gain, K_2, such that

$\int x_1^2 dt$ is minimal

Optimisation

- Based on Ricatti equations
  - LQR in 20-sim or Matlab
  - only for quadratic criteria
- Hill climbing
  - systematic search method
  - e.g. 20-sim
  - any well chosen criterion
- Hill climbing
  - find the top of an unknown hill in the fog

20-sim



Demo's
20-sim:
K1, K2
K1 fixed
optimise K2

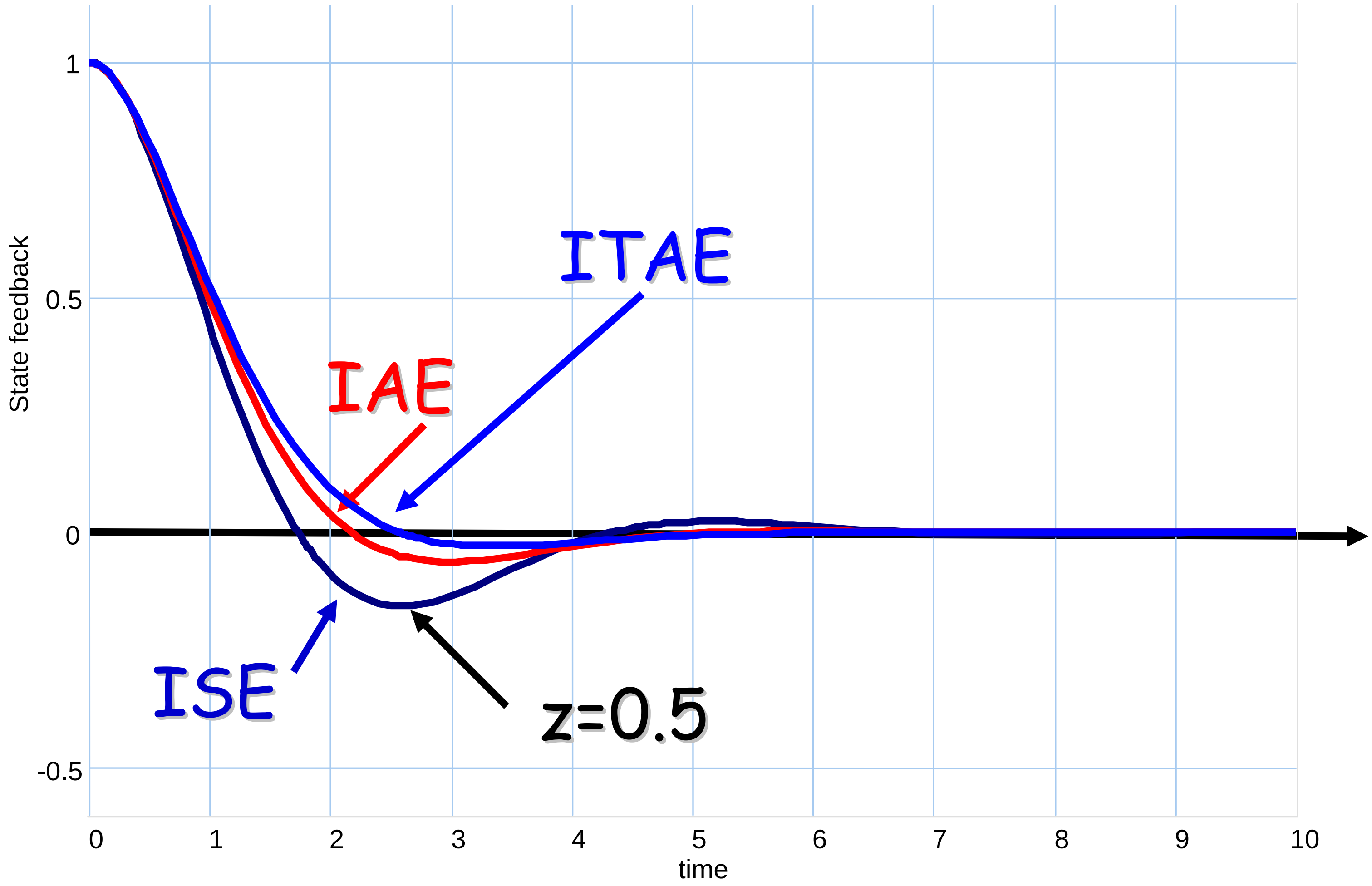
Demo: state feedback optimization K1 fixed

Demo: state feedback optimization

Criteria

ISE	$\int e^2\,dt$	more weight of large errors
IAE	$\int \lvert e \rvert\,dt$	
ITAE	$\int \lvert e \rvert\,t\,dt$	more weight on steady-state errors

Responses



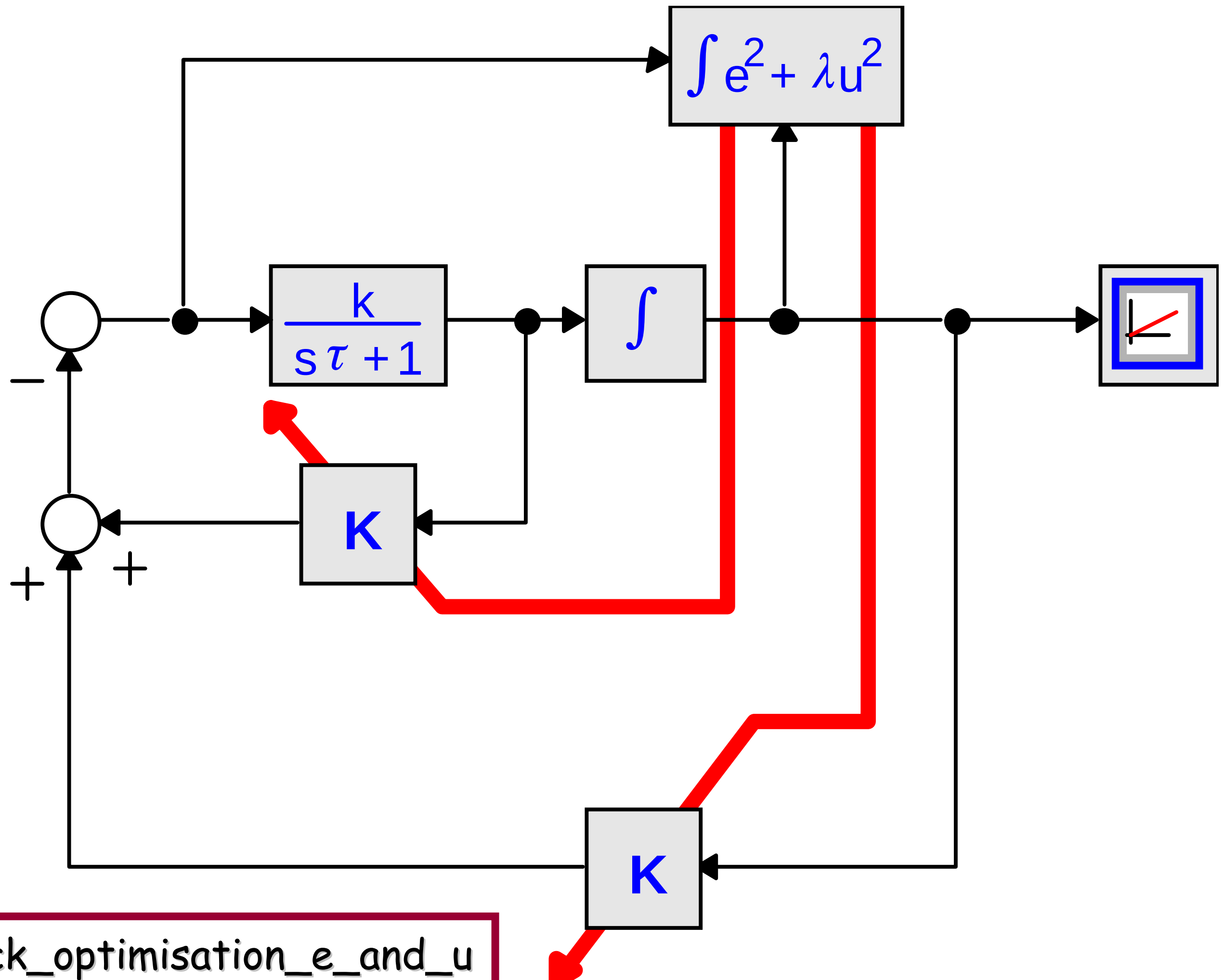
Weighting x and u

- When we weight both x and u, all feedback gains may be optimised simultaneously

$$J = \int \left(e^2 + \lambda u^2 \right) dt$$

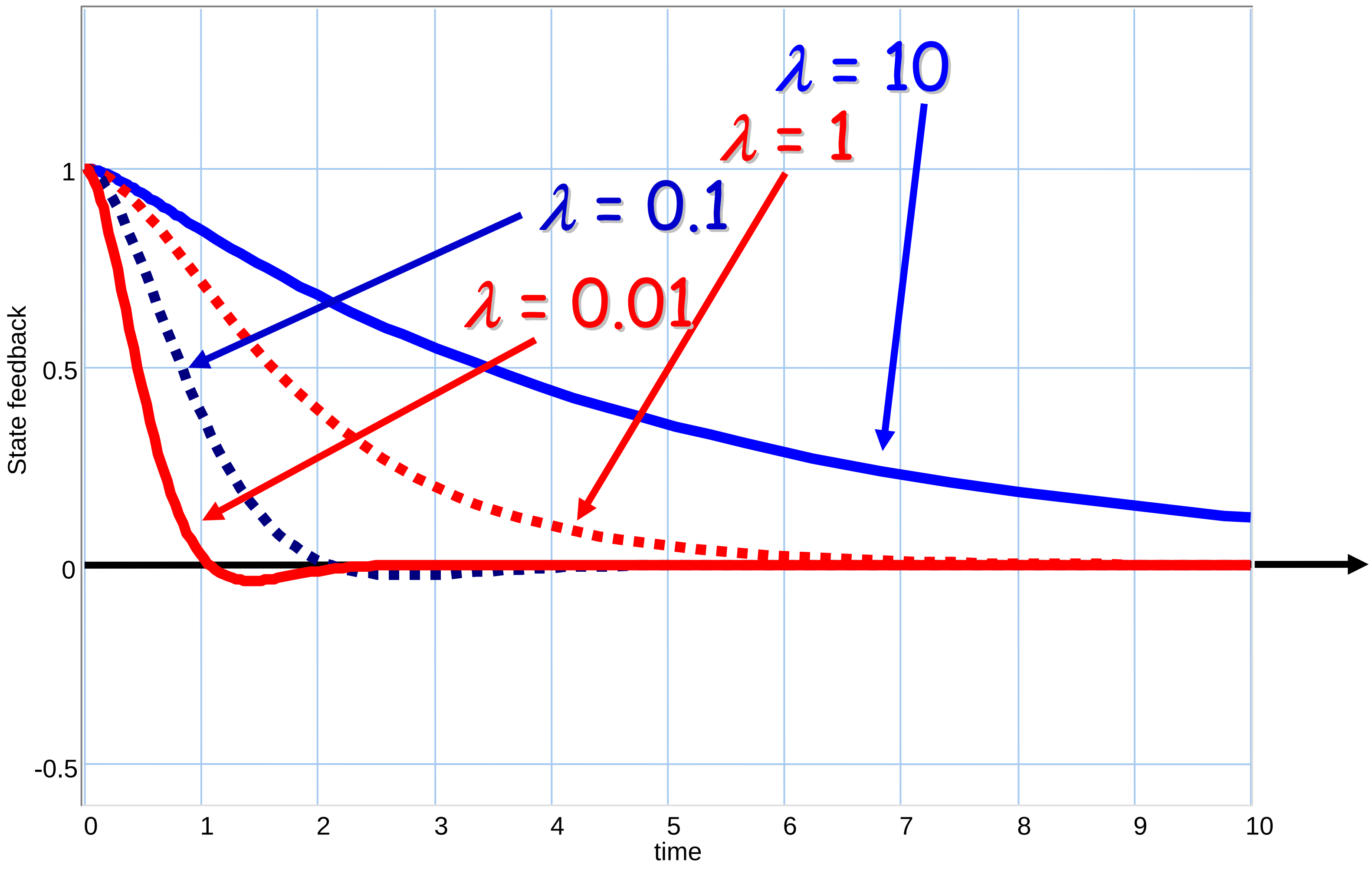
more weight on λ, leads to smaller u, and slower response

Weighting x and u

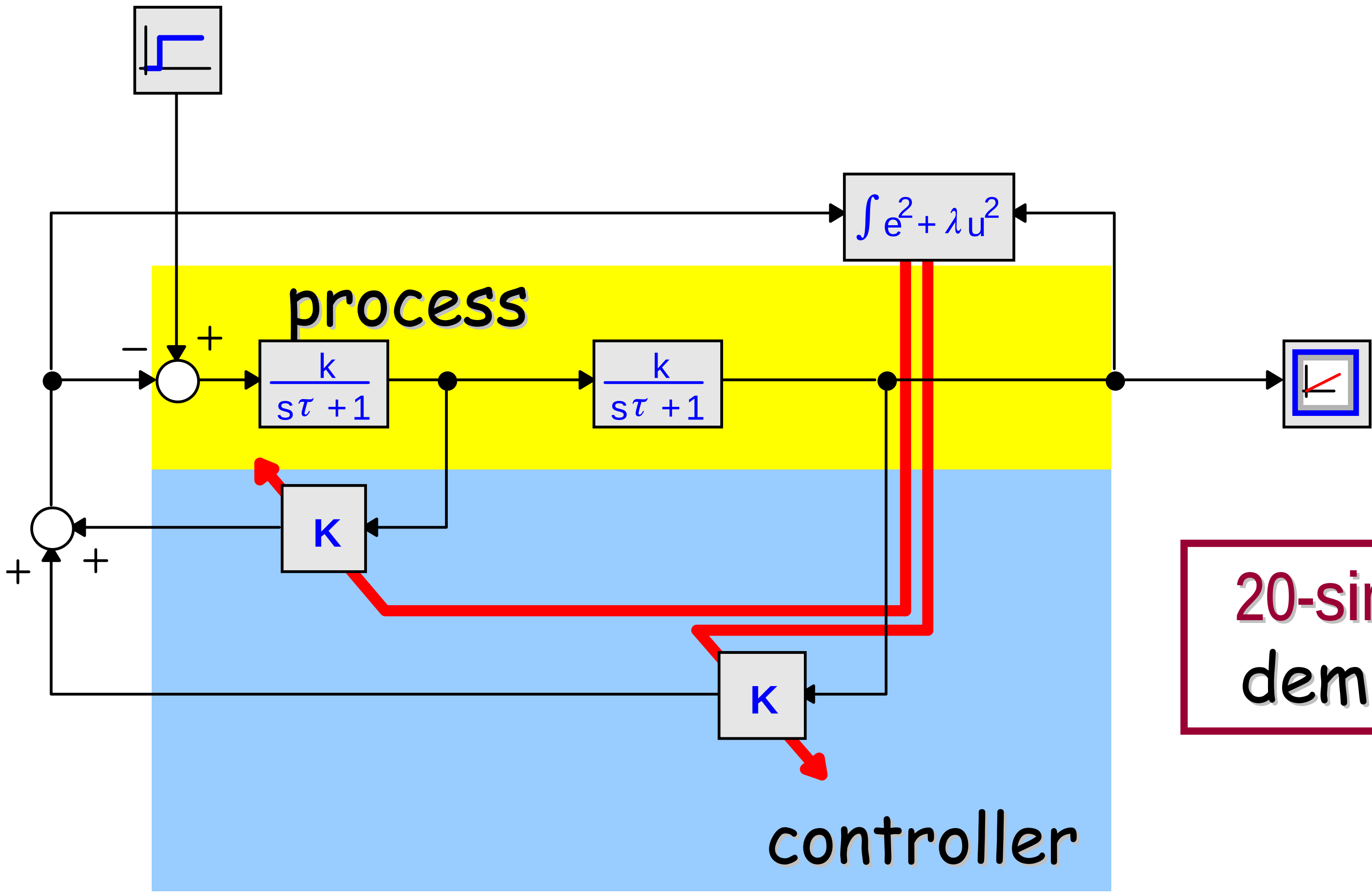


Demo: State_feedback_optimisation_e_and_u

Responses

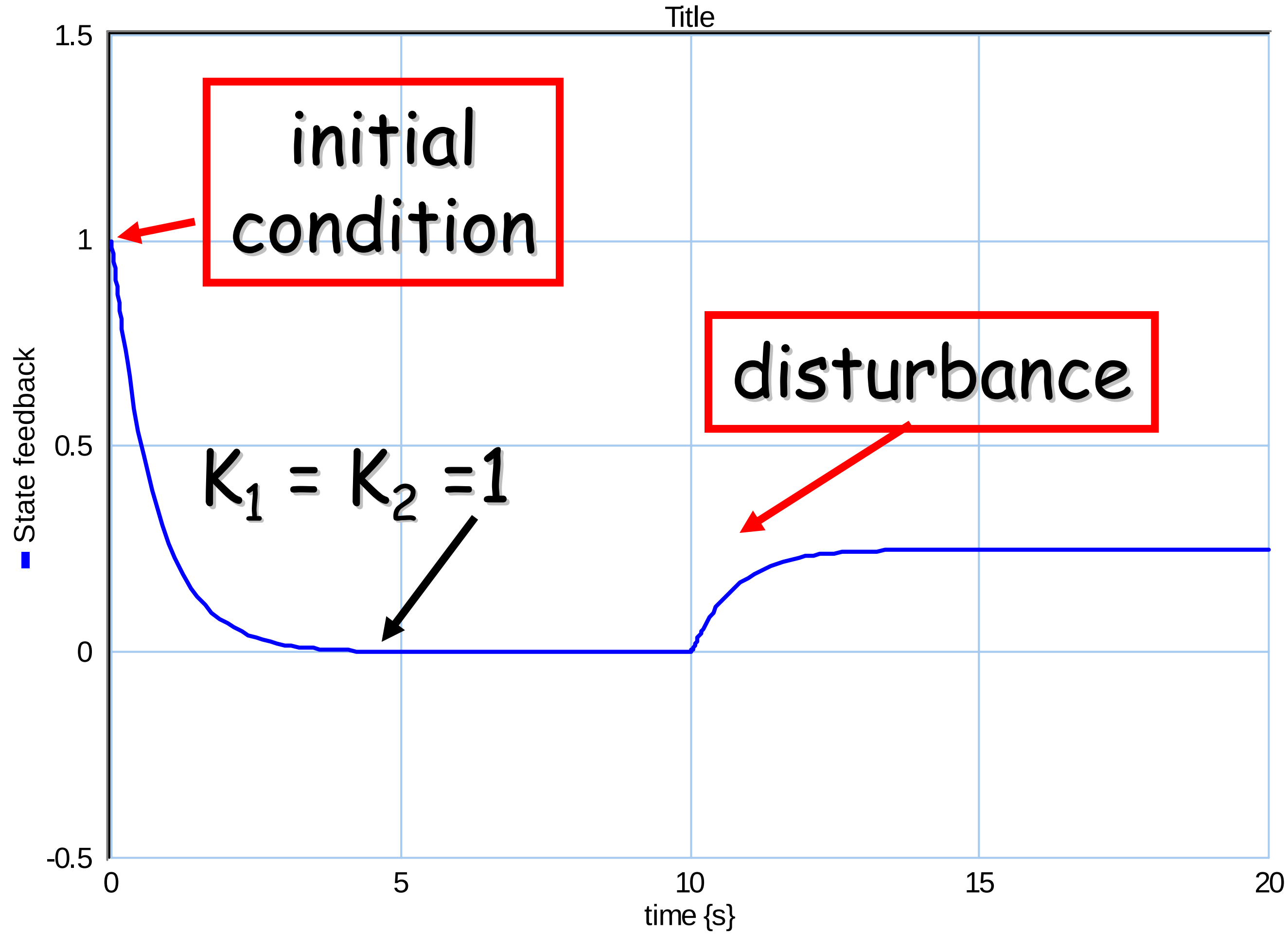


Type 0 system

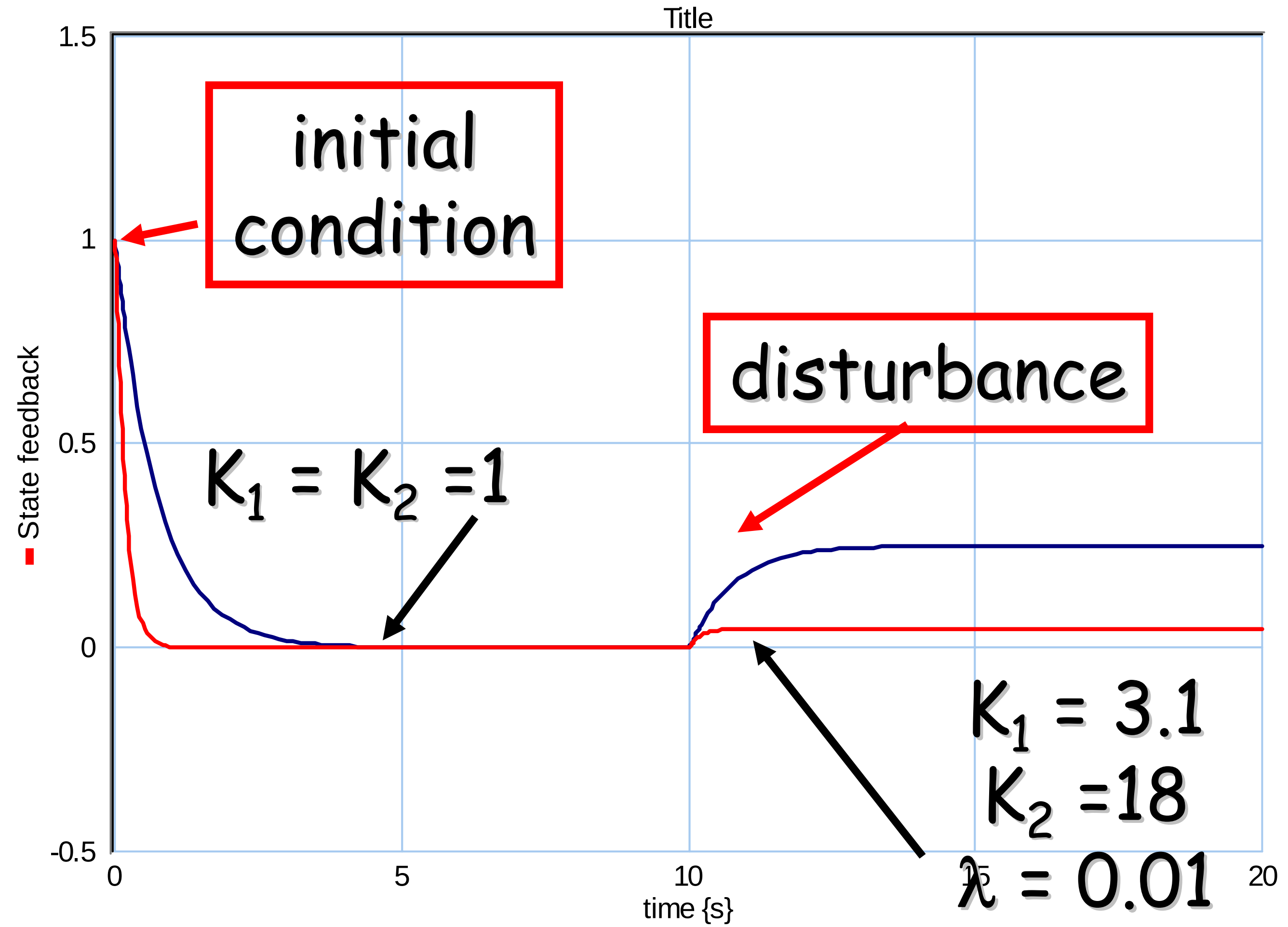


20-sim
demo

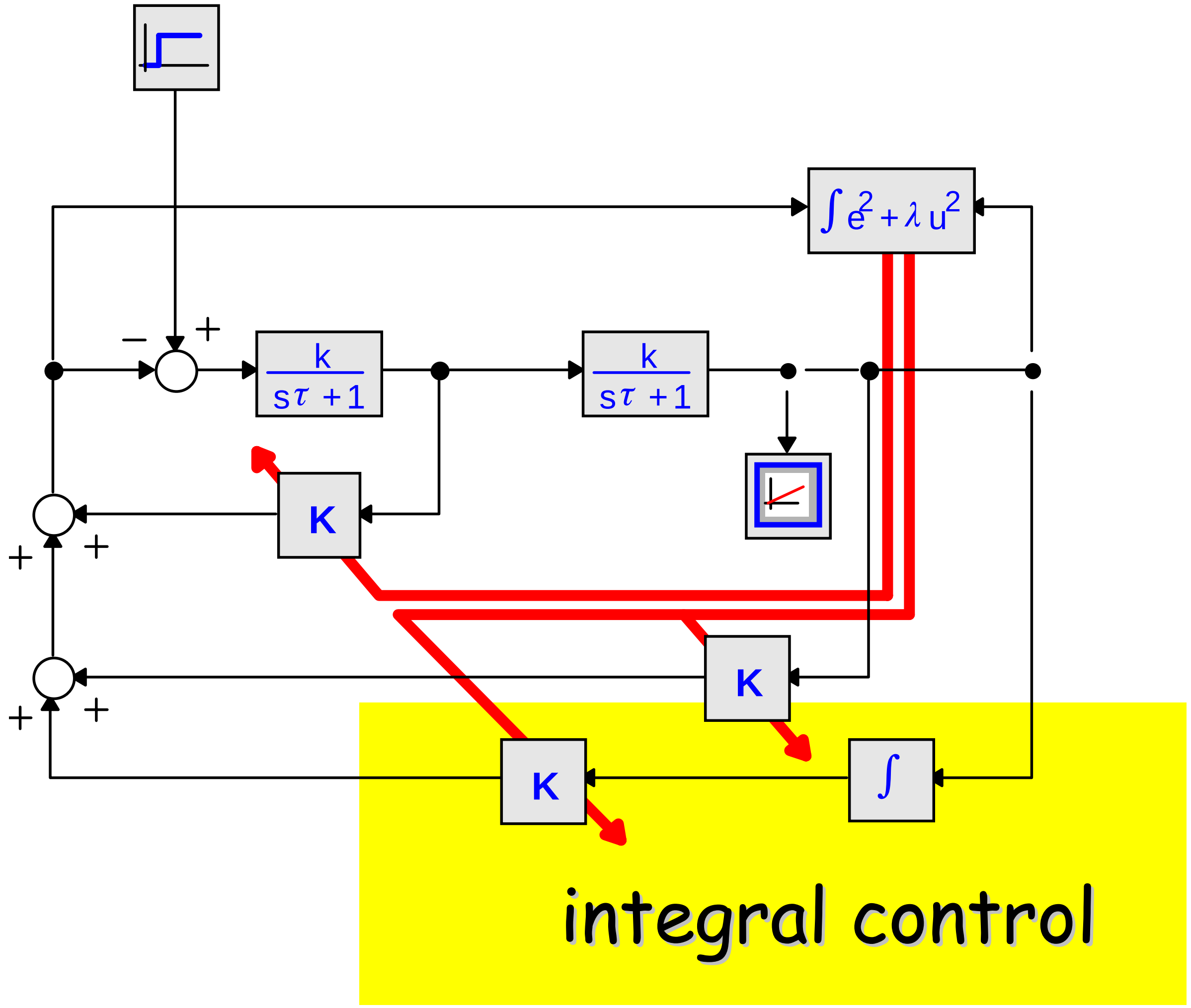
Responses


Title
1.5
1
0.5
0
-0.5
0
5
10
15
20
time {s}
State feedback
initial condition
disturbance
K1 = K2 =1

Responses

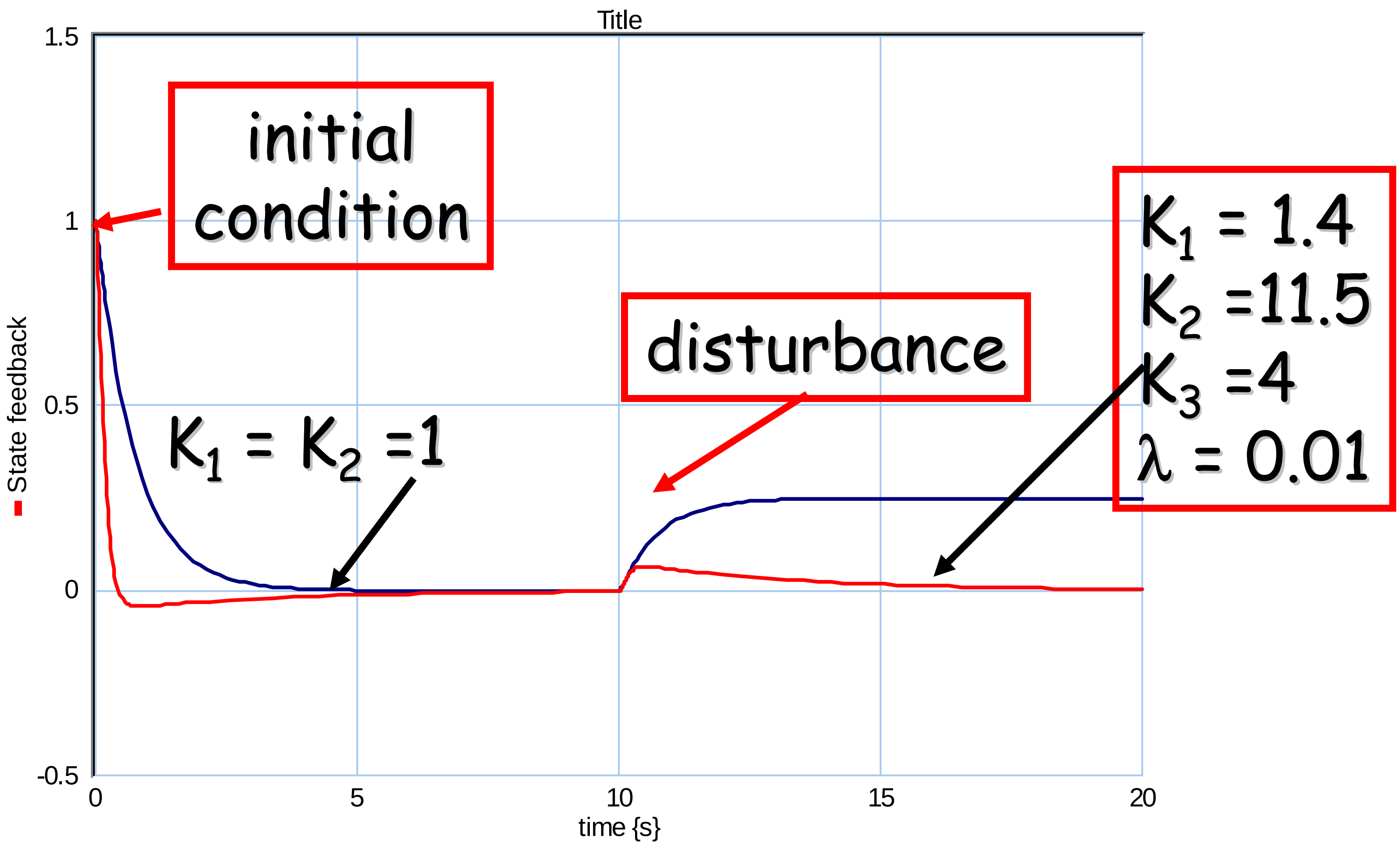

Title
1.5
1
0.5
0
-0.5
State feedback
0
5
10
15
20
time {s}
initial condition
disturbance
K1 = K2 =1
K1 = 3.1
K2 =18
λ = 0.01

‘PID’-control

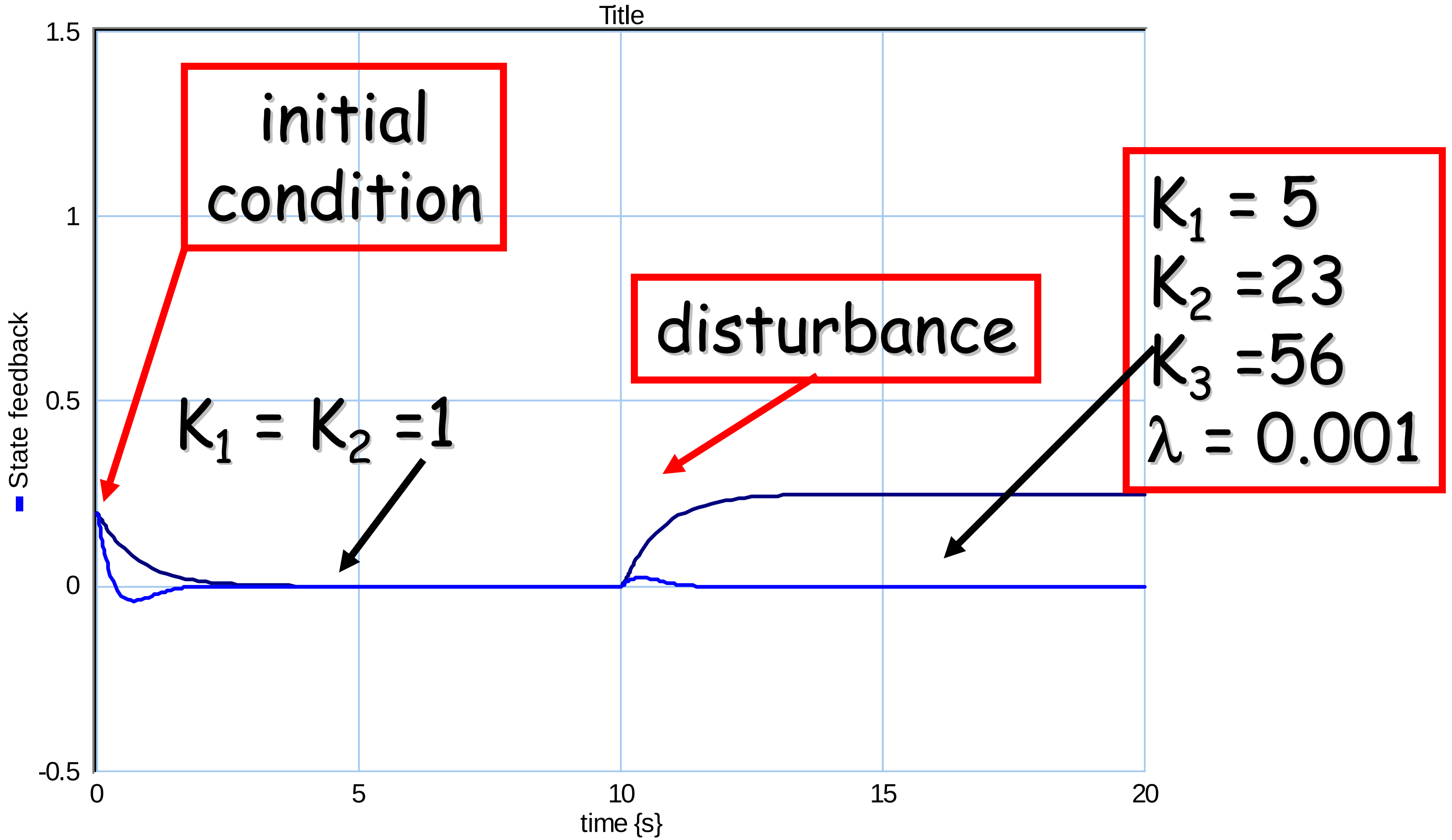


20-sim
demo

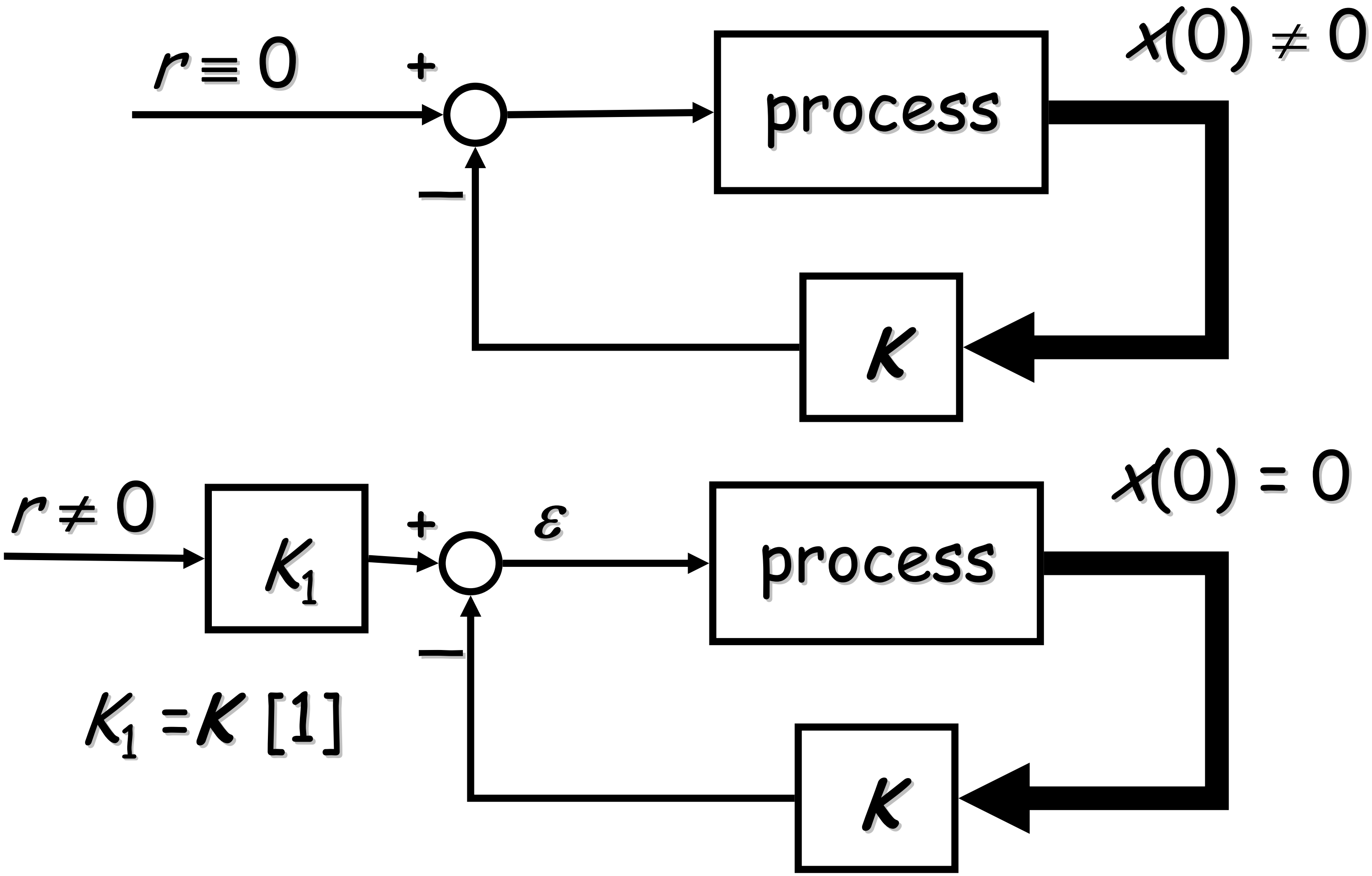
Responses


Title
1.5
1
0.5
0
-0.5
0
5
10
15
20
time {s}
State feedback
initial condition
disturbance
K1 = K2 =1
K1 = 1.4
K2 =11.5
K3 =4
λ = 0.01

Responses (2)

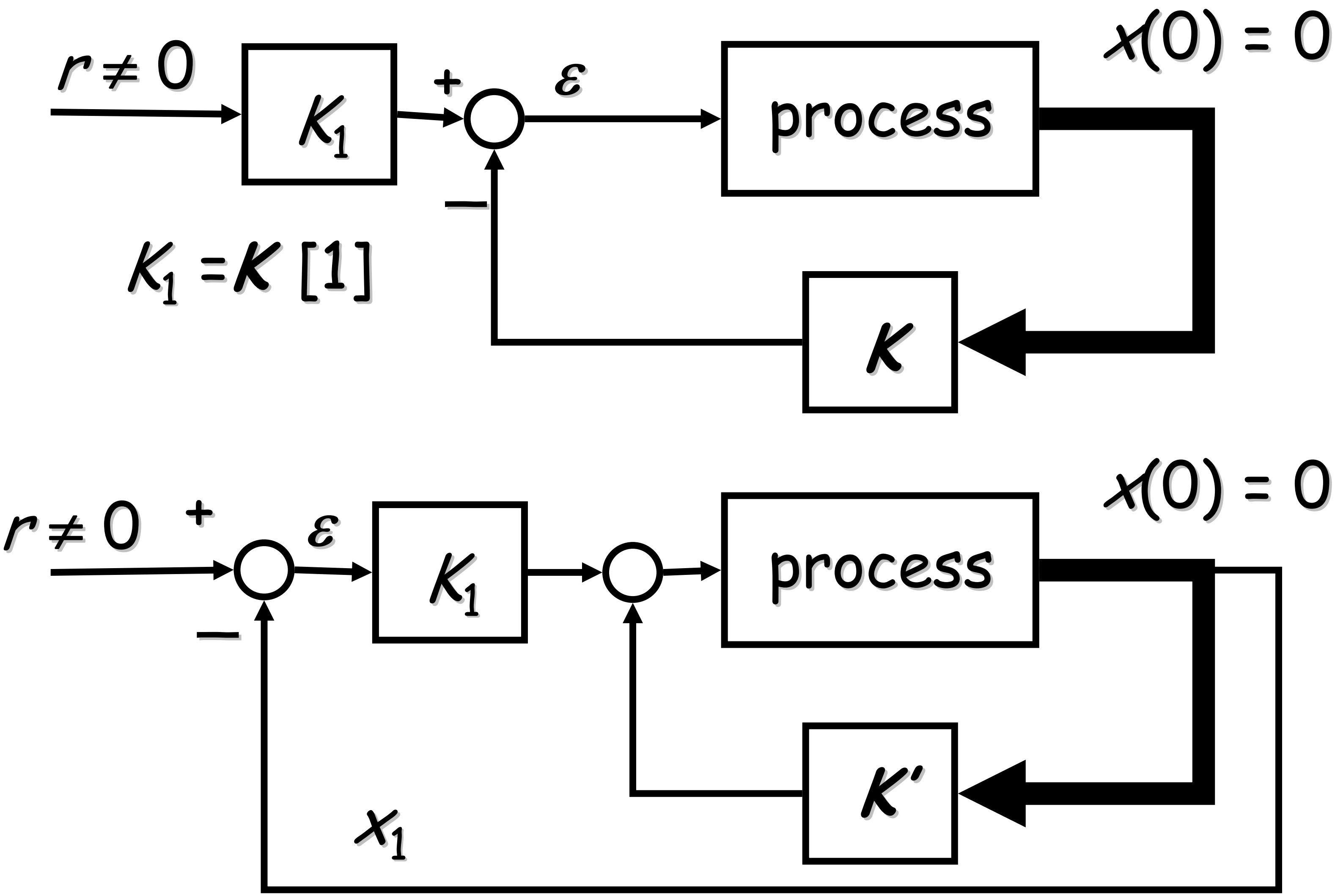

Title
1.5
1
0.5
0
-0.5
0
5
10
15
20
time {s}
State feedback
initial condition
disturbance
$K_1 = K_2 = 1$
$K_1 = 5$
$K_2 = 23$
$K_3 = 56$
$\lambda = 0.001$

Reference ≠ 0

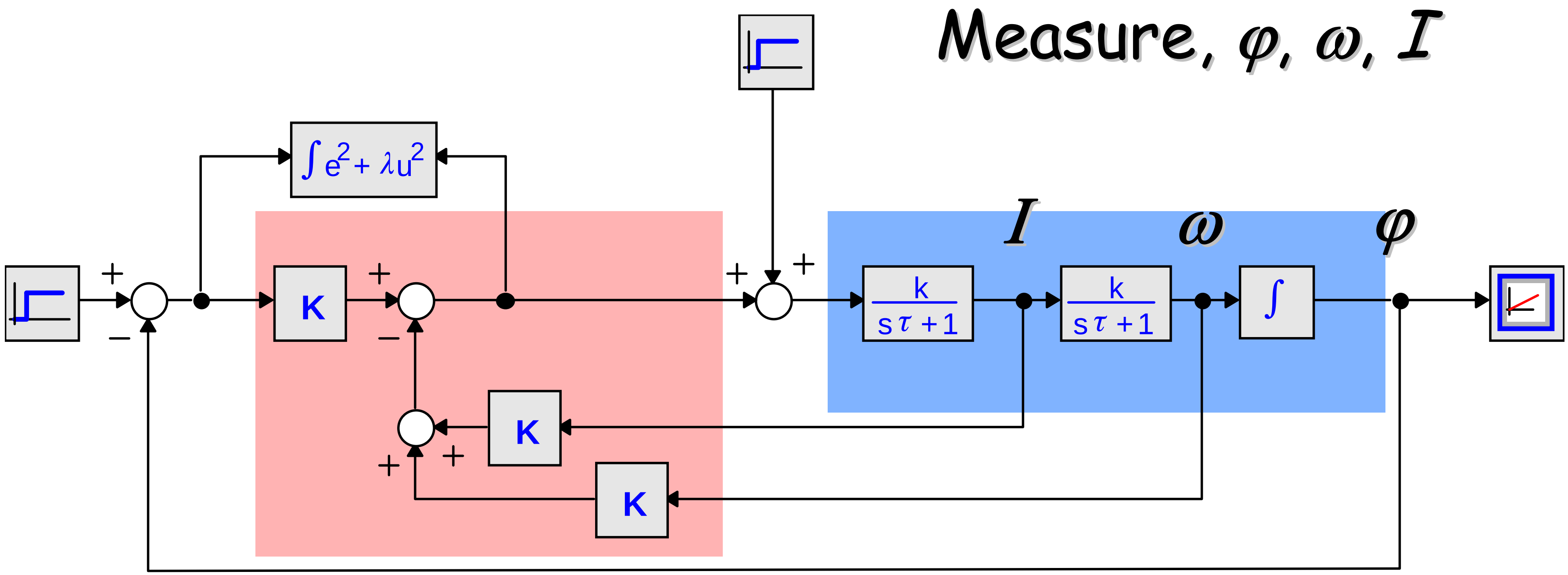


$K_1 = K\,[1]$

Reference ≠ 0



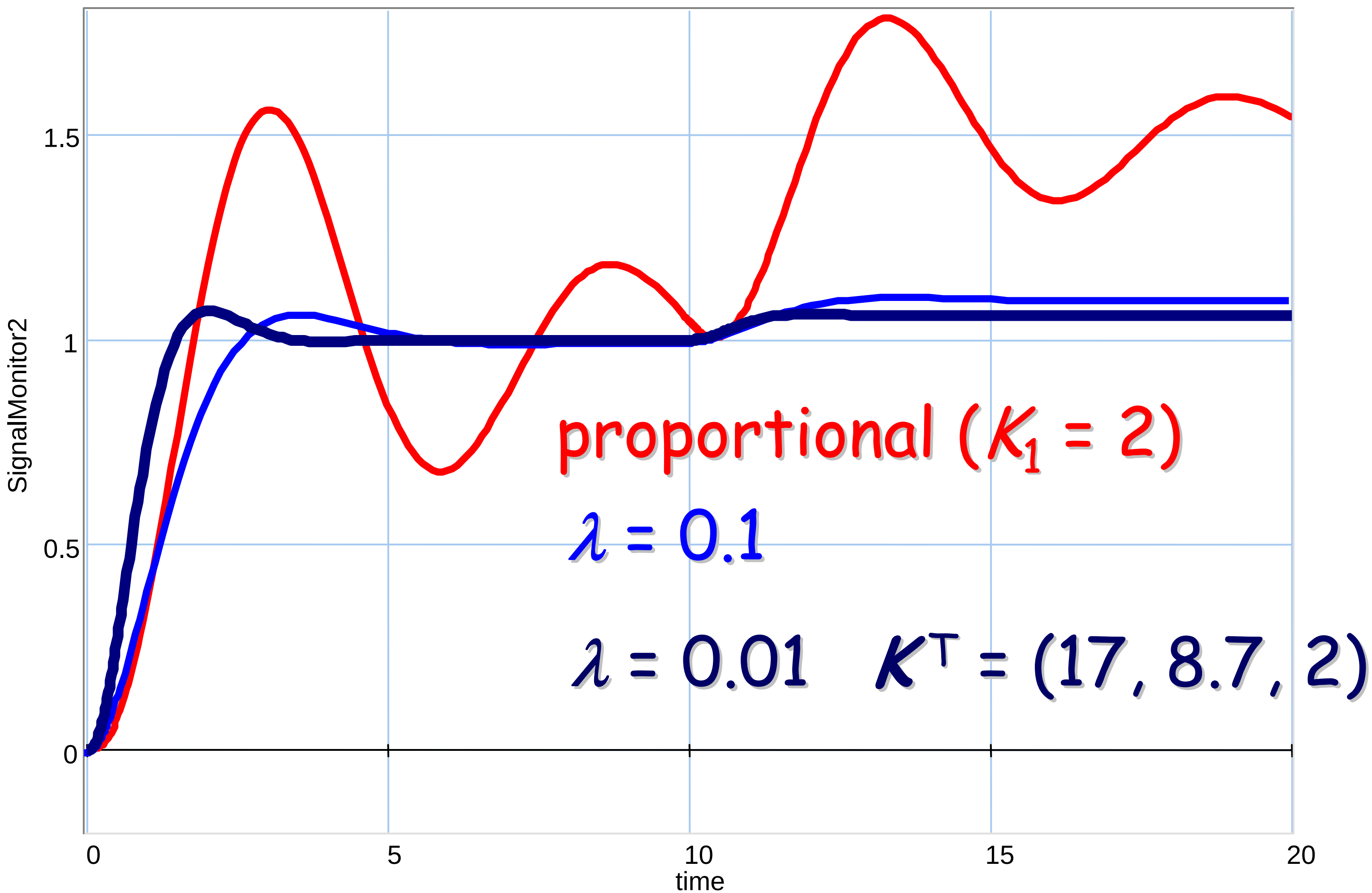
Example robot link



Exercise: do this optimisation yourself (process parameters as in sheet 3 of s-plane design)

20-sim

Responses


1.5
1
0.5
0
SignalMonitor2
0
5
10
15
20
time
proportional (K_1 = 2)
λ = 0.1
λ = 0.01 K^T = (17, 8.7, 2)

Conclusions

- State feedback
  - allows poles to placed at any desired location
  - specially suited for computer-supported design
  - requires that all states be available
  - this is not always the case
  - may require state estimation