Block Diagrams, Signal Flow Graphs, Sensitivity

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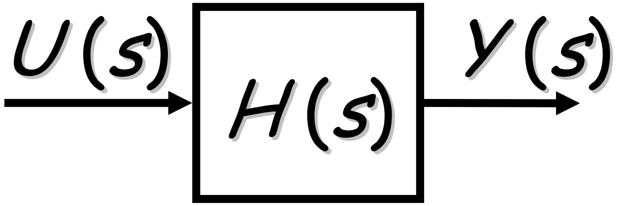
Contents

- Block diagrams
    - rules
- Signal flow graphs
    - Mason's rule
- Sensitivity
    - sensitivity
    - complementary sensitivity
    - steady state errors
    - system type (versus system order)

Block diagrams (basics)



$$Y(s) = H(s)U(s)$$
$$Y = HU$$

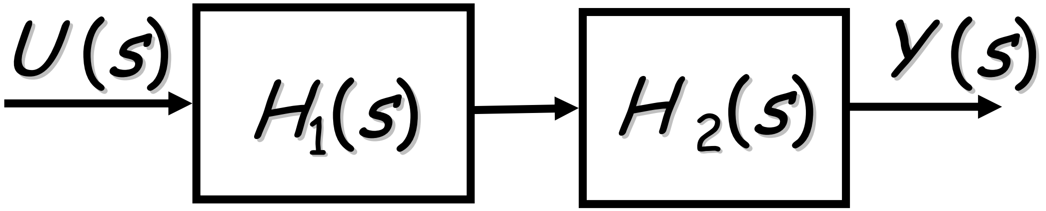
- Linear systems
- Signals:
  - Laplace transformations
- Contents of blocks:
  - transfer functions

lower case

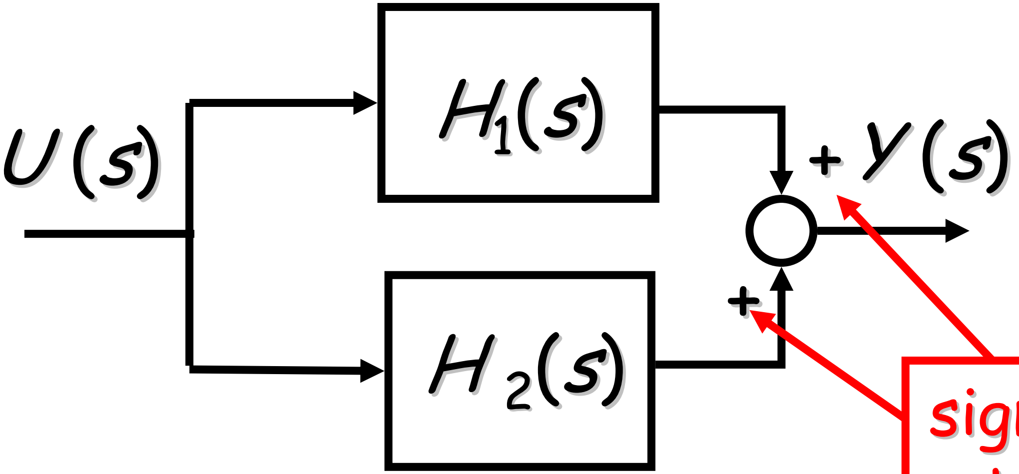
$y(t)$ versus $Y(s)$

capital font

Block diagrams (connections)



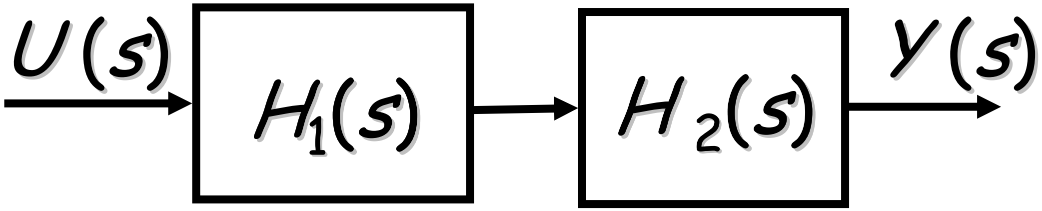
$$Y = H_1 H_2 U$$



$$Y = H_1U + H_2U$$
$$= (H_1 + H_2)U$$

signs always at the left of arrows

Example (series)



$$Y = (H_1 H_2) U$$

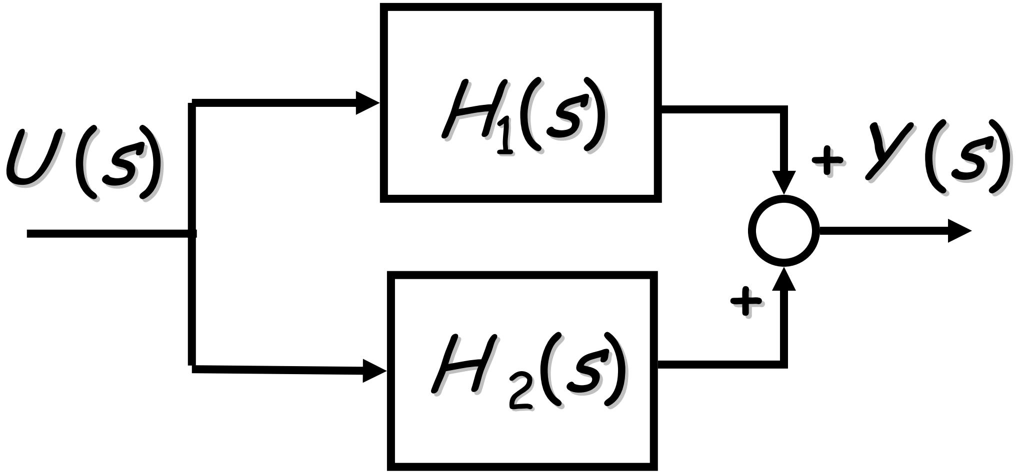
$$H_1 = \frac{K_1}{s\tau_1 + 1}$$

$$H_2 = \frac{K_2}{s\tau_2 + 1}$$

$$H_1 H_2 = \frac{K_1}{s\tau_1 + 1} \frac{K_2}{s\tau_2 + 1}$$

$$= \frac{K_1 K_2}{(s\tau_1 + 1)(s\tau_2 + 1)}$$

Example (parallel)



$$Y = H_1U + H_2U$$
$$= (H_1 + H_2)U$$

$$H_1 = \frac{K_1}{s\tau_1 + 1}$$

$$H_2 = \frac{K_2}{s\tau_2 + 1}$$

$$H_1 + H_2 = \frac{K_1}{s\tau_1 + 1} + \frac{K_2}{s\tau_2 + 1}$$

$$= \frac{K_1\left(s\tau_2 + 1\right) + K_2\left(s\tau_1 + 1\right)}{\left(s\tau_1 + 1\right)\left(s\tau_2 + 1\right)}$$

Example (parallel)

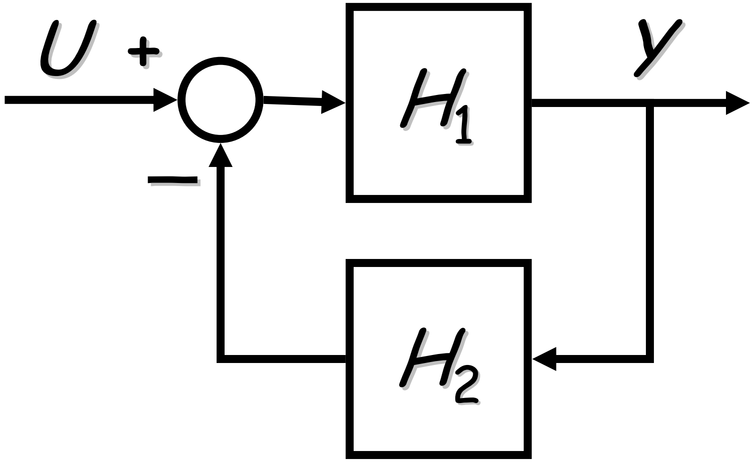
$$H_1+H_2=\frac{K_1\left(s\tau_2+1\right)+K_2\left(s\tau_1+1\right)}{\left(s\tau_1+1\right)\left(s\tau_2+1\right)}$$

$$=\frac{\left(sK_1\tau_2+K_1\right)+\left(sK_2\tau_1+K_2\right)}{\left(s\tau_1+1\right)\left(s\tau_2+1\right)}=\frac{s\left(K_1\tau_2+K_2\tau_1\right)+K_1+K_2}{\left(s\tau_1+1\right)\left(s\tau_2+1\right)}$$

$$=\left(K_1+K_2\right)\frac{s\left(\frac{K_1\tau_2+K_2\tau_1}{K_1+K_2}\right)+1}{\left(s\tau_1+1\right)\left(s\tau_2+1\right)}$$

zero

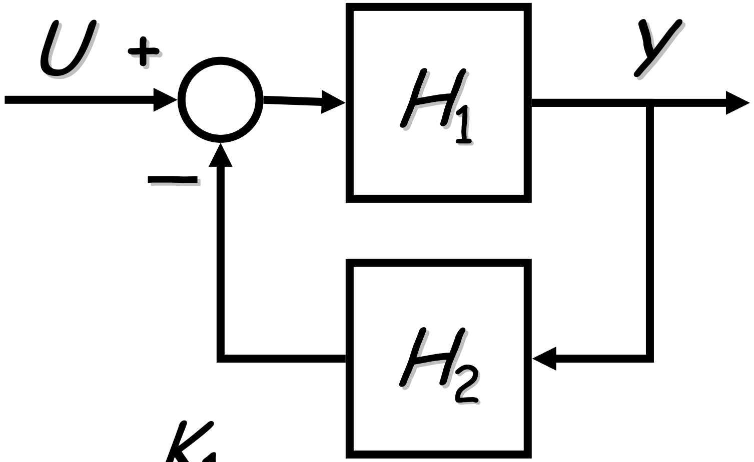
Block diagrams (feedback)



$$y = \frac{H_1}{1 + H_1 H_2} U$$

$$y = \frac{\text{forward path}}{1 - \text{loop transfer}} U$$

Example (feedback)



$$Y = HU$$

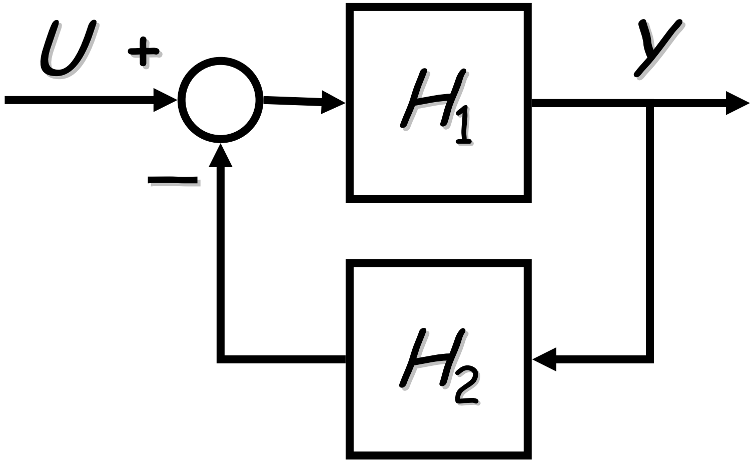
$$H = \frac{\dfrac{K_1}{s\tau_1+1}}{1+\dfrac{K_1}{s\tau_1+1}\dfrac{K_2}{s\tau_2+1}}$$

$$H_1 = \frac{K_1}{s\tau_1+1}$$

$$H_2 = \frac{K_2}{s\tau_2+1}$$

$$= \frac{K_1\left(s\tau_2+1\right)}{\left(s\tau_1+1\right)\left(s\tau_2+1\right)+K_1K_2}$$

Example (feedback)



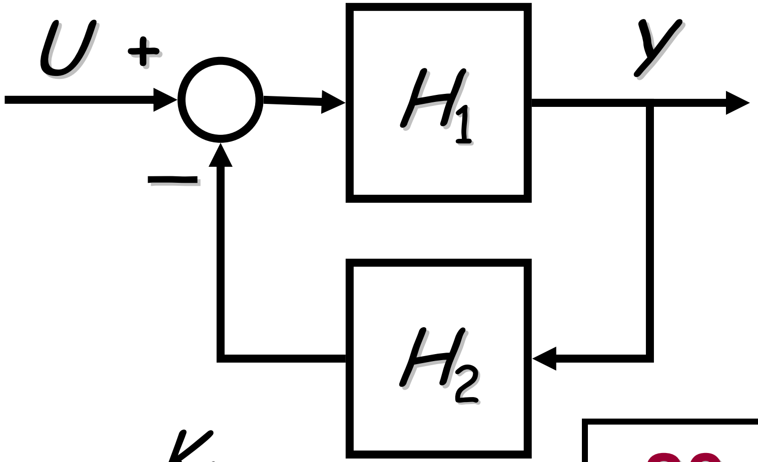
pole of H_2 is zero of H

$$H = \frac{K_1\left(s\tau_2 + 1\right)}{\left(s\tau_1 + 1\right)\left(s\tau_2 + 1\right) + K_1 K_2}$$

poles of H not equal to poles of H_1 and H_2

unless $K_1 = 0$ and/or $K_2 = 0$

Demo 20-sim



$$y = HU$$

$$H = \frac{\dfrac{K_1}{s+1}}{1 + \dfrac{K_1}{s+1}\dfrac{2}{s+2}}$$

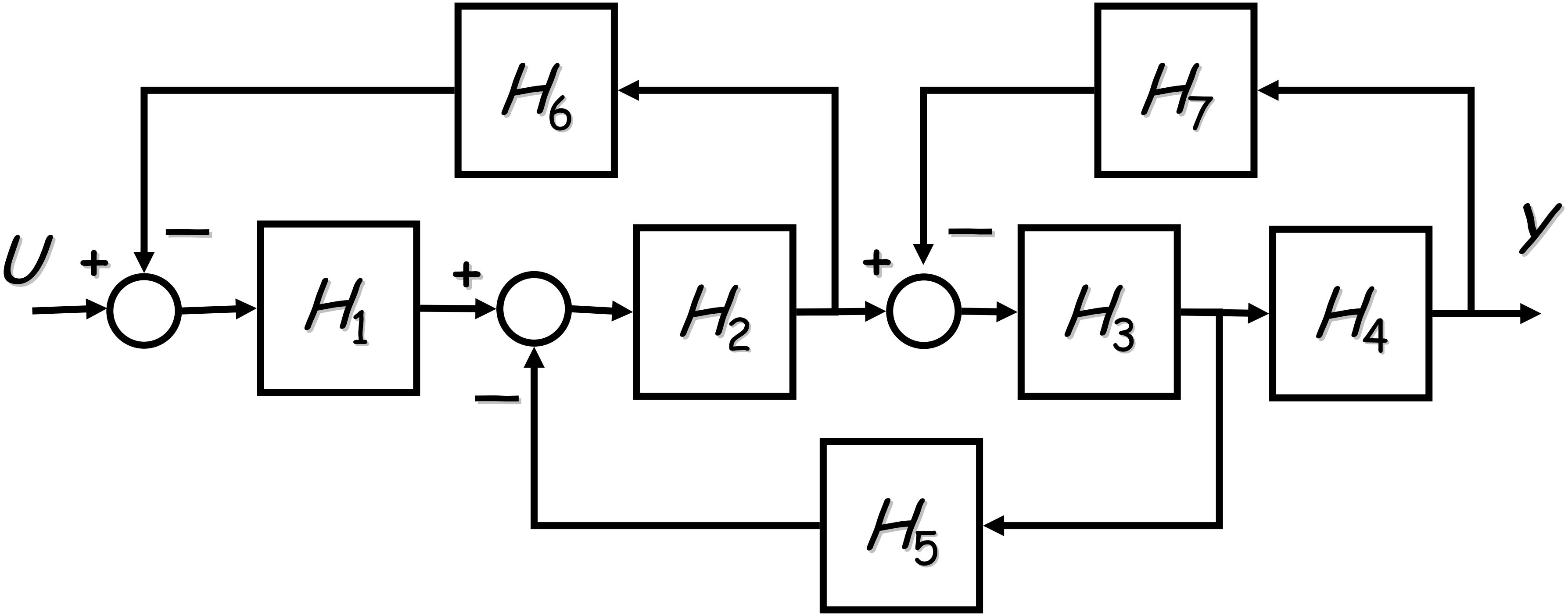
$$H_1 = \frac{K_1}{s+1}$$

20-sim
demo

$$H_2 = \frac{1}{0.5s+1} = \frac{2}{s+2}$$

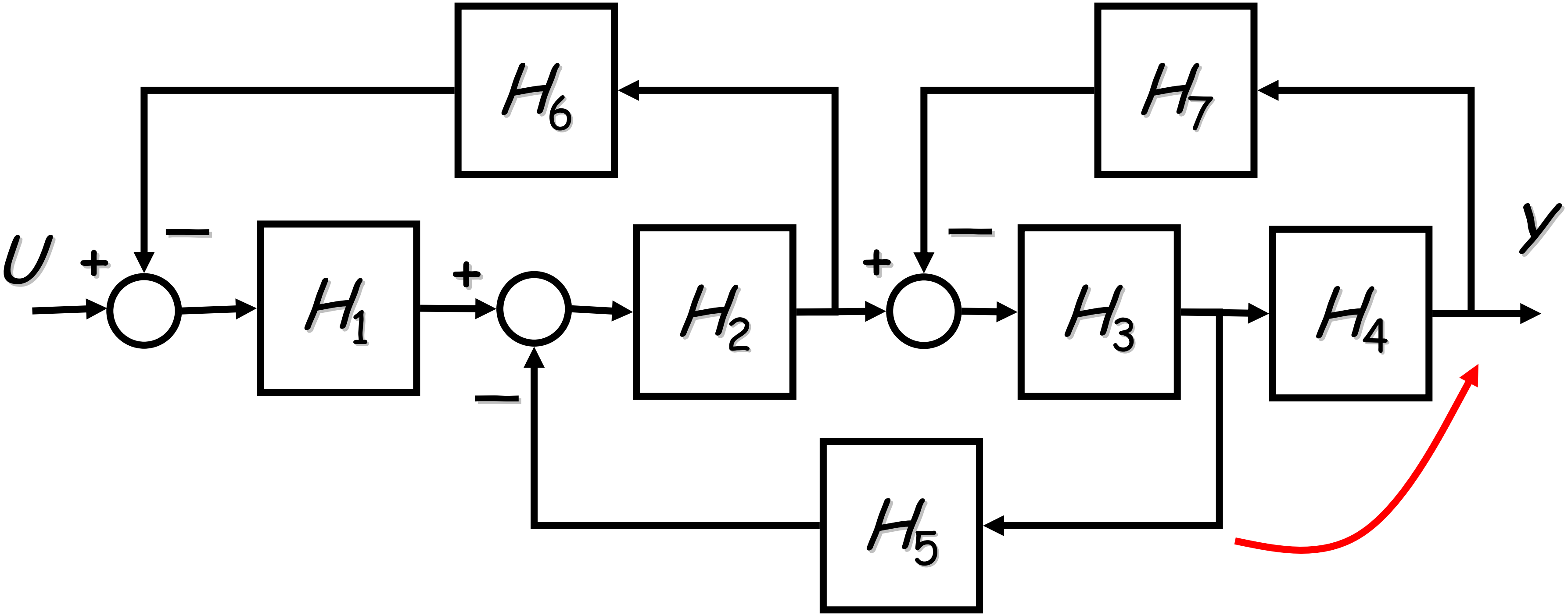
$$= \frac{K_1(0.5s+1)}{(s+1)(0.5s+1)+K_1}$$

Example (complex system)

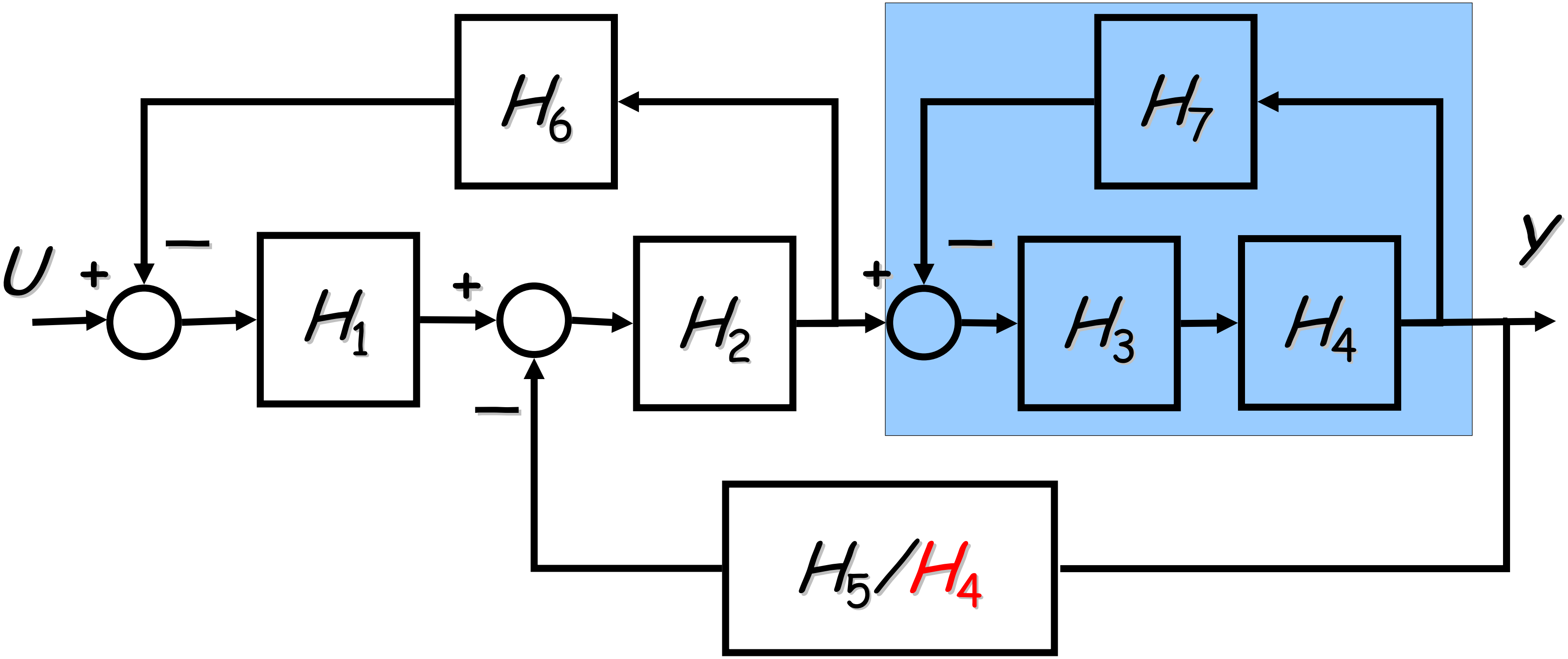


Exercise: Do it yourself

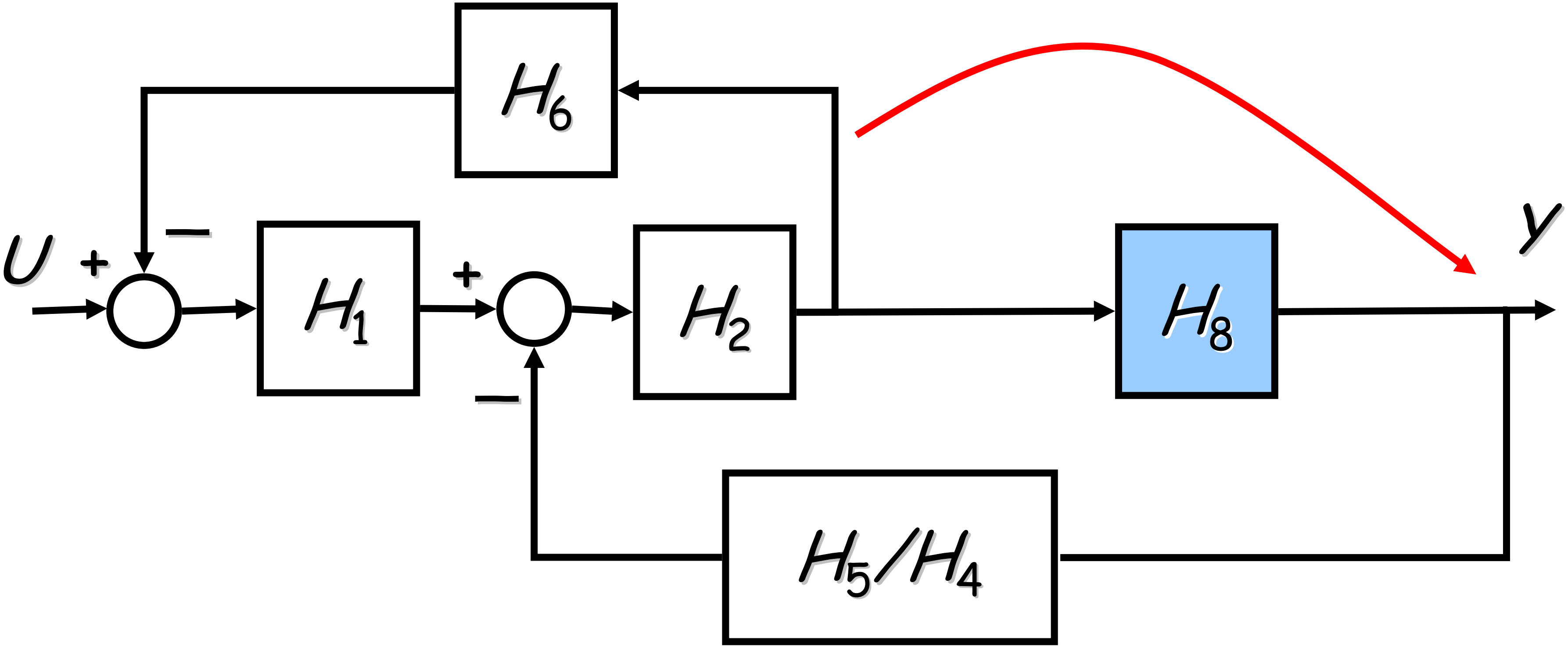
Example (complex system)



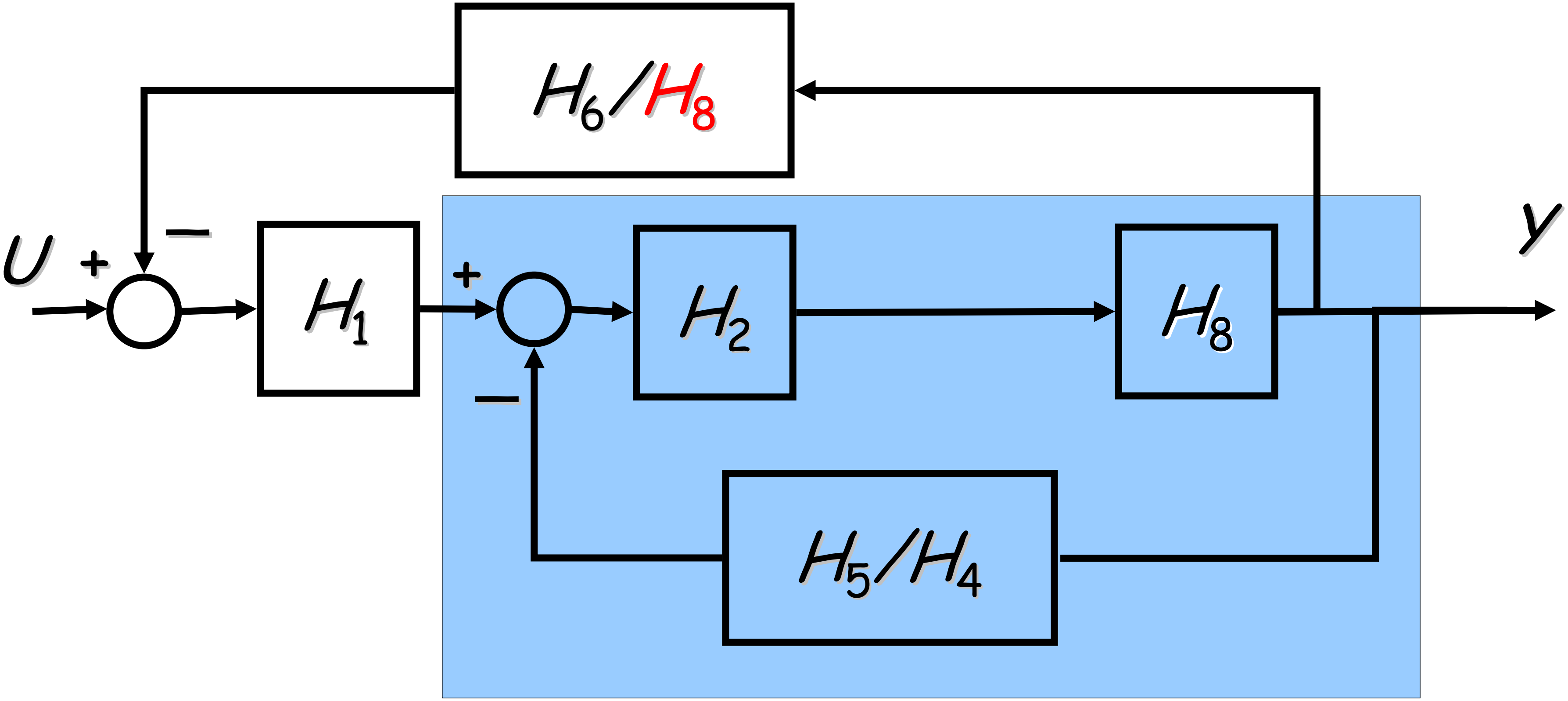
Example (complex system)



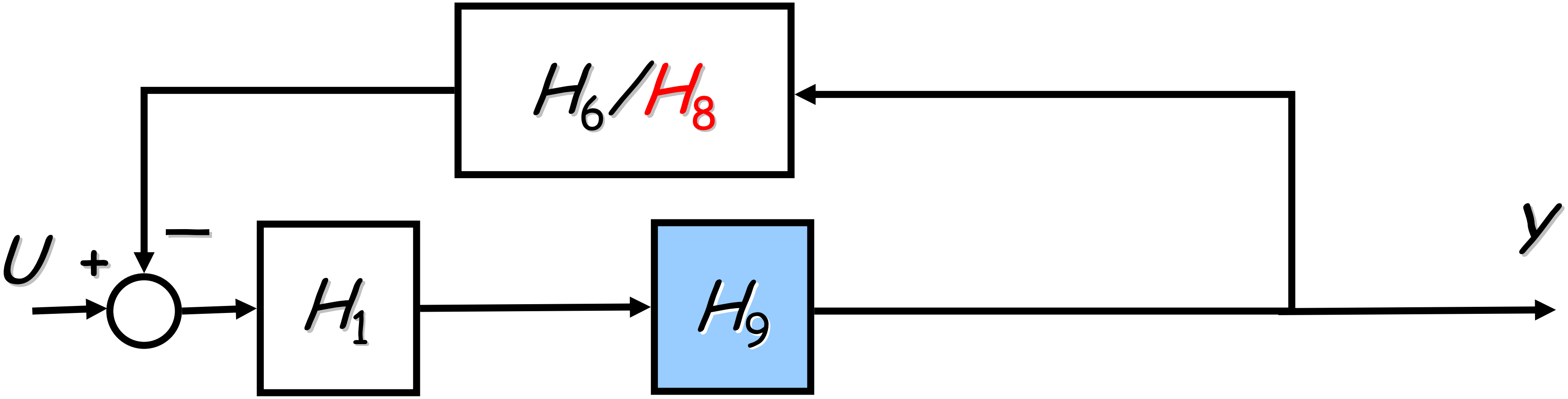
Example (complex system)

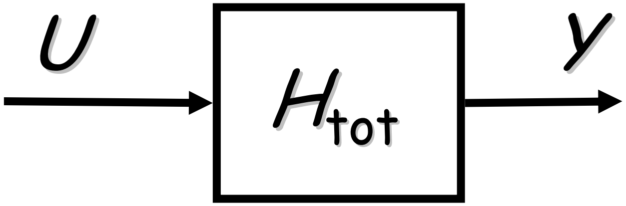


Example (complex system)



Example (complex system)

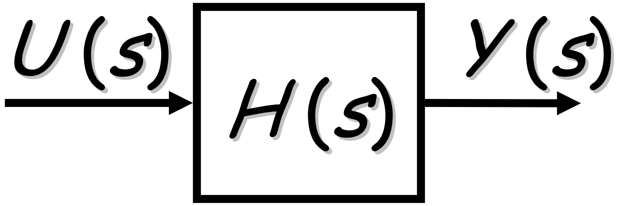




Mason’s rule

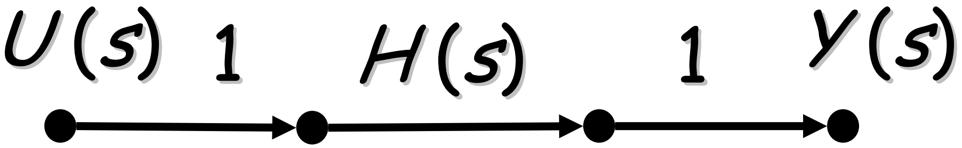
- To compute transfer functions of complex diagrams
- Mostly uses signal flow graphs
- With care it can be used with block diagrams as well

Signal Flow Graphs (basics)



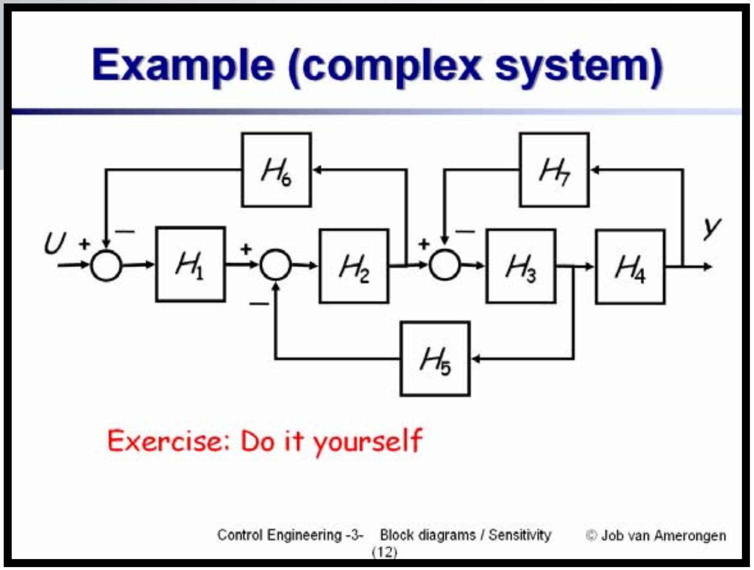
$$Y(s) = H(s)U(s)$$

$$Y = HU$$



- Linear systems
- Nodes:
  - signals
- Branches:
  - transfer functions

Example



Example (complex system)

Exercise: Do it yourself

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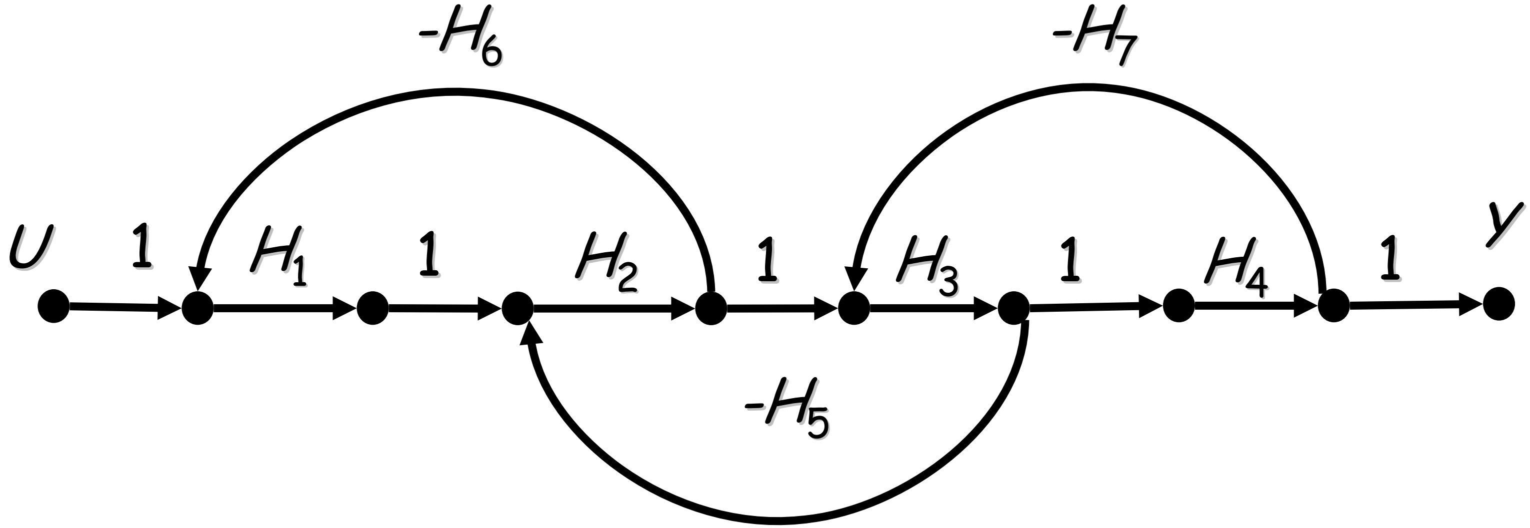
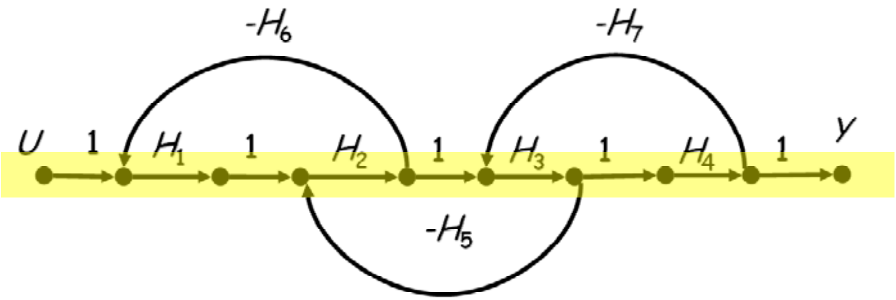


Definitions

- Path:
  - a succession of branches in the direction of the arrows that do not pass any node more than once

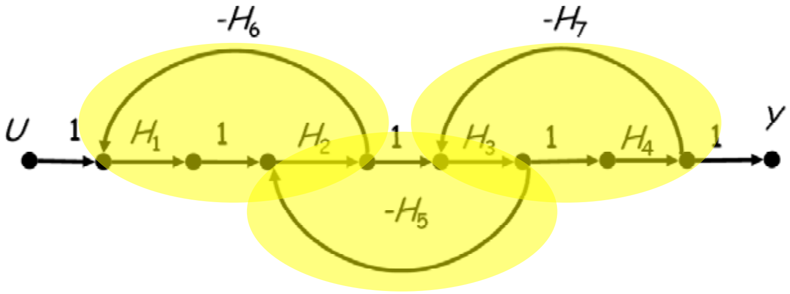


There is 1 path from U to Y

$H_1 H_2 H_3 H_4$ = path gain

Definitions

- Loop:
  - a closed succession of branches in the direction of the arrows that do not pass any node more than once



There are three loops

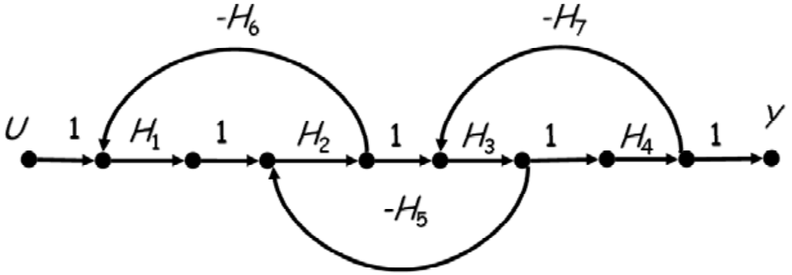
$-H_1 H_2 H_6$ = loop gain

$-H_3 H_4 H_7$ = loop gain

$-H_2 H_3 H_5$ = loop gain

Definitions

- Touching:
- Loops with one or more nodes in common are called touching. A loop and a path are touching when they have a common node.



Touching loops:

$-H_1 H_2 H_6$ with $-H_2 H_3 H_5$

$-H_3 H_4 H_7$ with $-H_2 H_3 H_5$

Non-touching loops:

$-H_1 H_2 H_6$ and $-H_3 H_4 H_7$

Definitions

- **Determinant:**
  - The determinant Δ of a signal flow graph is:

Δ = 1 -(sum of al loop gains)

+ (sum of products of gains of all possible combinations of 2 non-touching loops)

- (sum of products of gains of all possible combinations of 3 non-touching loops)

+ ...

Definitions

- **Cofactor**:
  - The cofactor of the *i*-th path, denoted by Δ_i, is the determinant of the signal flow graph formed by deleting all loops touching path *i*.

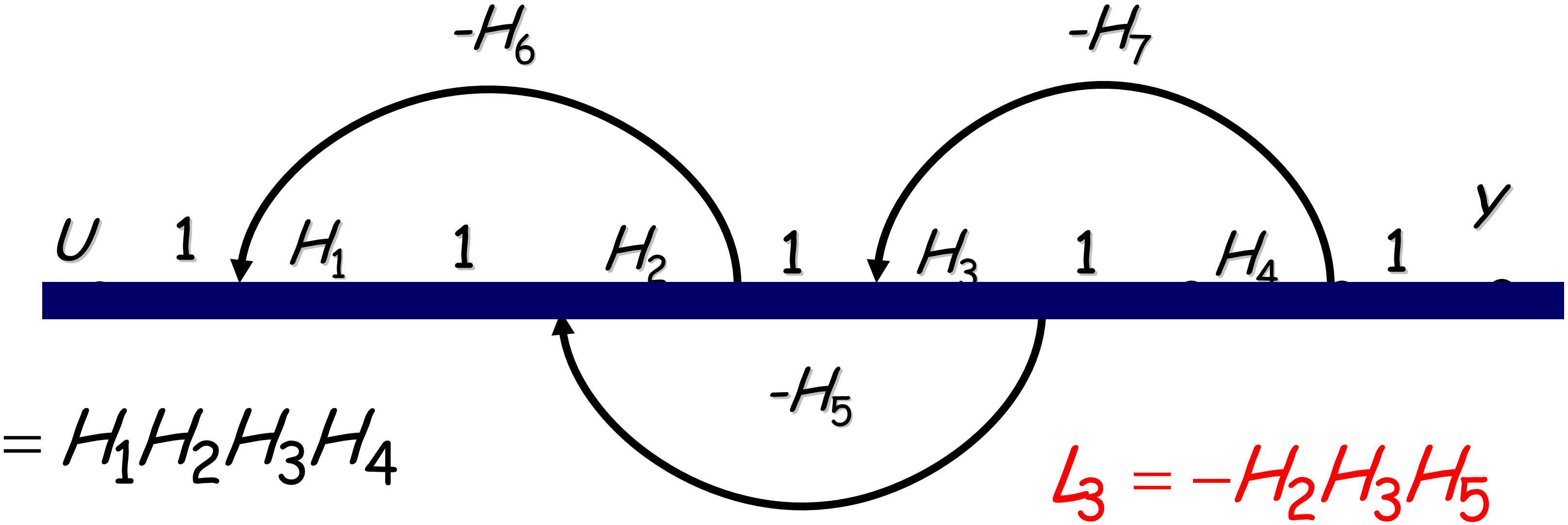
Mason’s rule

$$H(s) = \frac{P_1\Delta_1 + P_2\Delta_2 + \ldots}{\Delta}$$

Example

$L_1 = -H_1H_2H_6$

$L_2 = -H_3H_4H_7$



$P_1 = H_1H_2H_3H_4$

$L_3 = -H_2H_3H_5$

$\Delta_1 = 1$

(cofactor = 1)

$$\Delta = 1 + H_1H_2H_6 + H_3H_4H_7 + H_2H_3H_5$$
$$+H_1H_2H_6H_3H_4H_7$$

Example

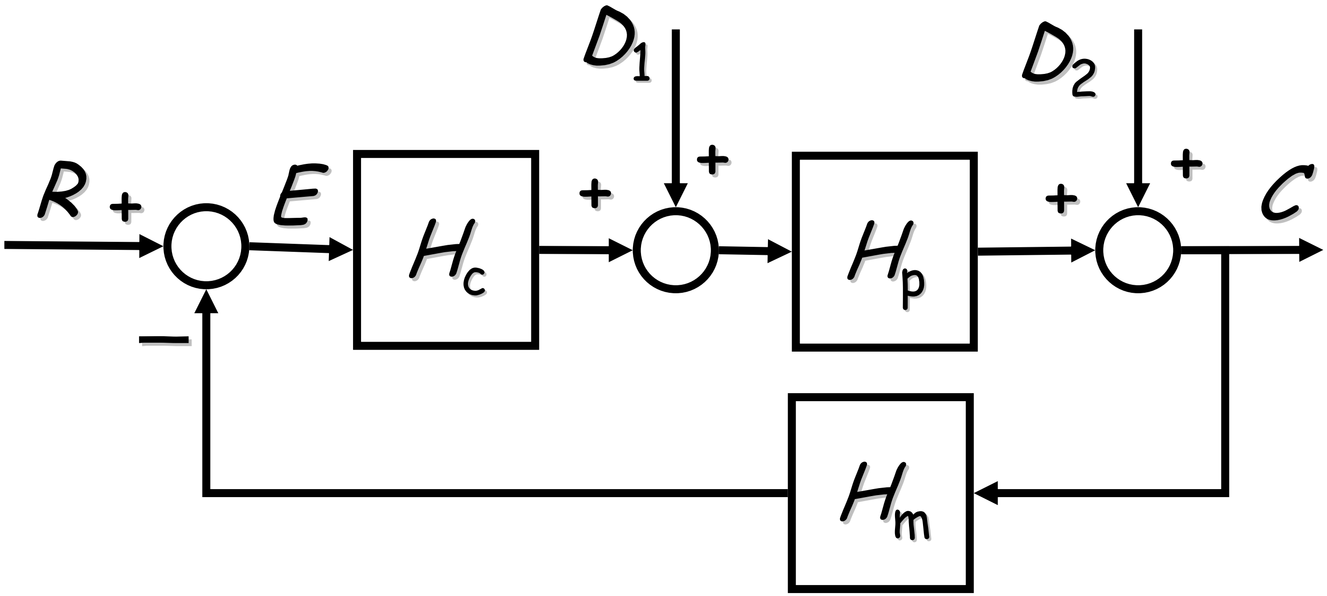
$$P_1 = H_1H_2H_3H_4$$

$$\Delta_1 = 1$$

$$\Delta = 1 + H_1H_2H_6 + H_3H_4H_7 + H_2H_3H_5$$
$$+H_1H_2H_6H_3H_4H_7$$

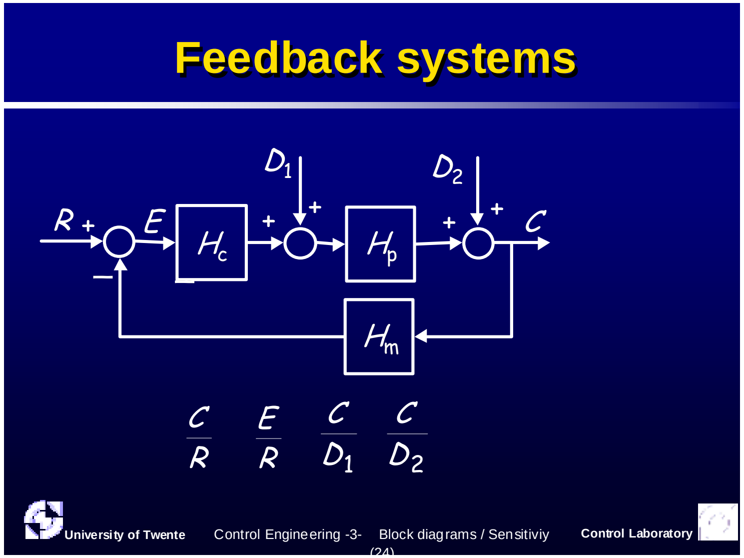
$$H = \frac{H_1H_2H_3H_4}{1 + H_1H_2H_6 + H_3H_4H_7 + H_2H_3H_5 + H_1H_2H_6H_3H_4H_7}$$

Feedback systems



$$\frac{C}{R} \qquad \frac{E}{R} \qquad \frac{C}{D_1} \qquad \frac{C}{D_2}$$

C/R



Feedback systems

$$\frac{C}{R} \quad \frac{E}{R} \quad \frac{C}{D_1} \quad \frac{C}{D_2}$$

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$$\frac{C}{R} = \frac{H_c H_p}{1 + H_c H_p H_m}$$

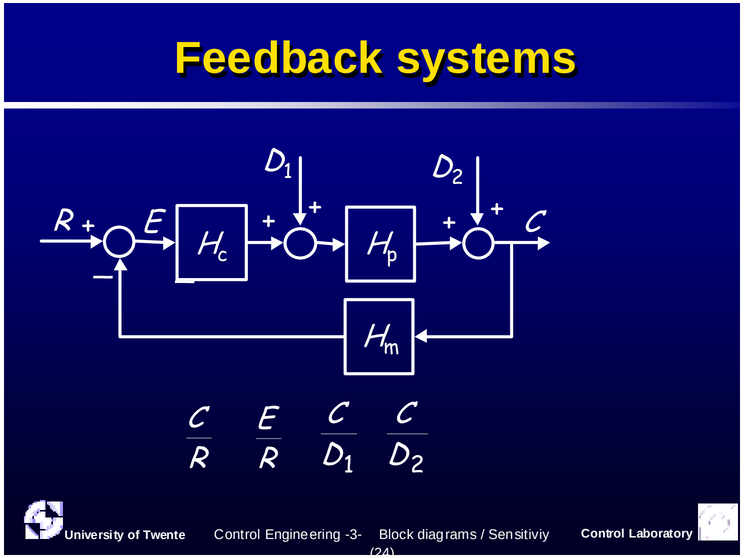
$$\text{Desired:} \quad \frac{C}{R} = 1$$

$$\longrightarrow \boxed{H_c} H_p \to \infty$$

$$\frac{C}{R} \approx \frac{H_c H_p}{H_c H_p H_m} = \frac{1}{H_m}$$

It is important that H_m is constant and in this configuration H_m should be 1

E/R



Feedback systems

$$\frac{C}{R} \quad \frac{E}{R} \quad \frac{C}{D_1} \quad \frac{C}{D_2}$$

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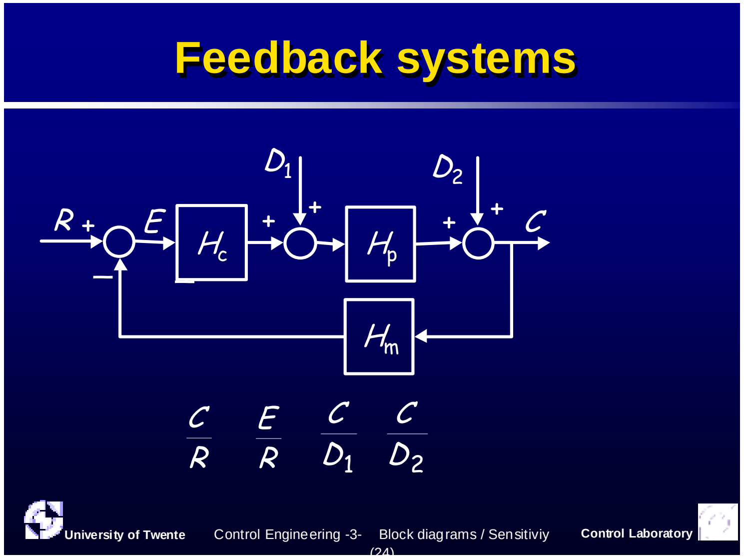
$$\frac{E}{R} = \frac{1}{1 + H_c H_p H_m} = \frac{1}{1 + H_L}$$

Desired: $\frac{E}{R} = 0$

$$\longrightarrow \quad H_L = H_c H_p H_m \to \infty$$

$$\frac{E}{R} \approx \frac{1}{\infty} \to 0$$

C/D_1



Feedback systems

$\frac{C}{R}$ $\frac{E}{R}$ $\frac{C}{D_1}$ $\frac{C}{D_2}$

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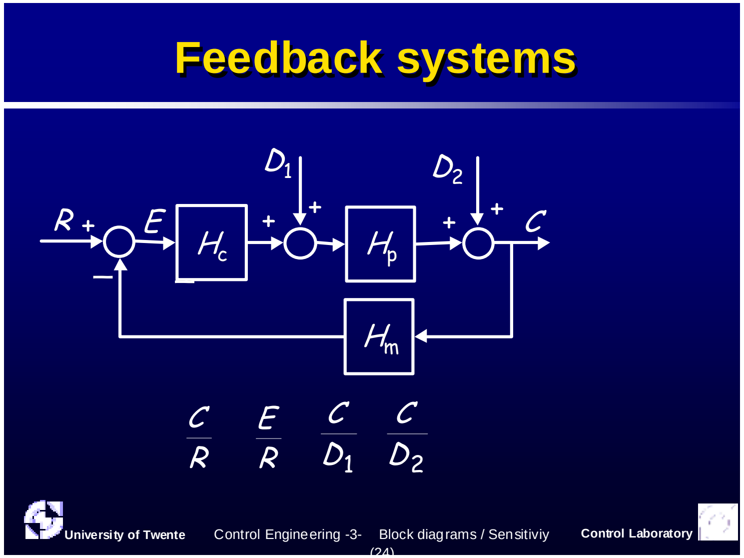
$$\frac{C}{D_1} = \frac{H_p}{1 + H_c H_p H_m}$$

Desired: $\frac{C}{D_1} = 0$

$$\longrightarrow \quad H_c H_m \to \infty$$

$$\frac{C}{D_1} \approx \frac{H_p}{H_c H_p H_m} = \frac{1}{H_c H_m} \to 0$$

C/D$_2$



Feedback systems

$\frac{C}{R}$ $\frac{E}{R}$ $\frac{C}{D_1}$ $\frac{C}{D_2}$

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$$\frac{C}{D_2} = \frac{1}{1 + H_c H_p H_m} = \frac{1}{1 + H_L}$$

Desired: $\frac{C}{D_2} = 0$

$$\longrightarrow \boxed{H_c} H_m H_p \to \infty$$

$$\frac{C}{D_2} \approx \frac{1}{H_c H_p H_m} \to 0$$

Conclusion

$$\frac{C}{R} = 1 \longrightarrow H_c H_p \to \infty$$

$$\frac{E}{R} = 0 \longrightarrow H_c H_p H_m \to \infty$$

$$\frac{C}{D_1} = 0 \longrightarrow H_c H_m \to \infty$$

$$\frac{C}{D_2} = 0 \longrightarrow H_c H_p H_m \to \infty$$

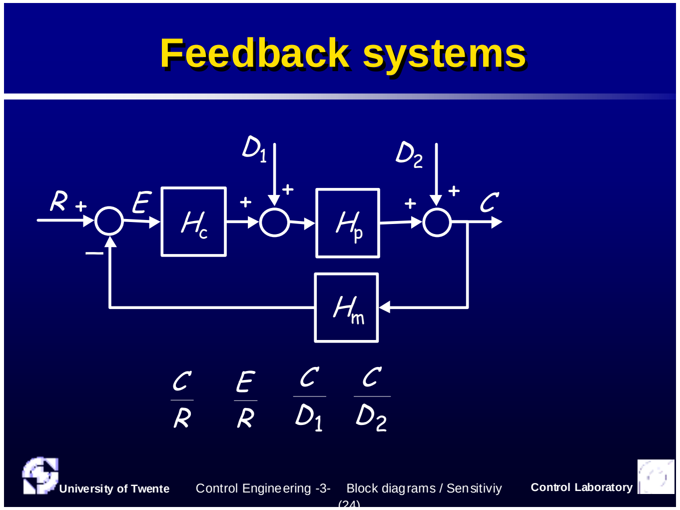
All goals can be achieved when

$$\boxed{H_c \to \infty}$$

⚠

stability

$H_{sensor} = 1$



Feedback systems

$\frac{C}{R}$ $\frac{E}{R}$ $\frac{C}{D_1}$ $\frac{C}{D_2}$

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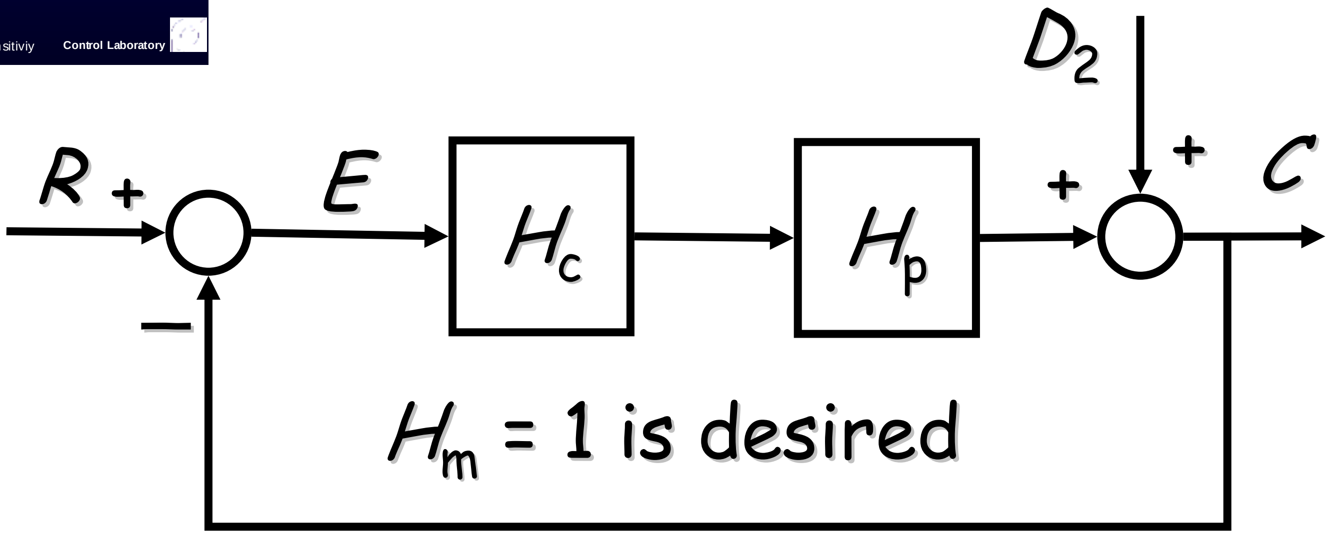
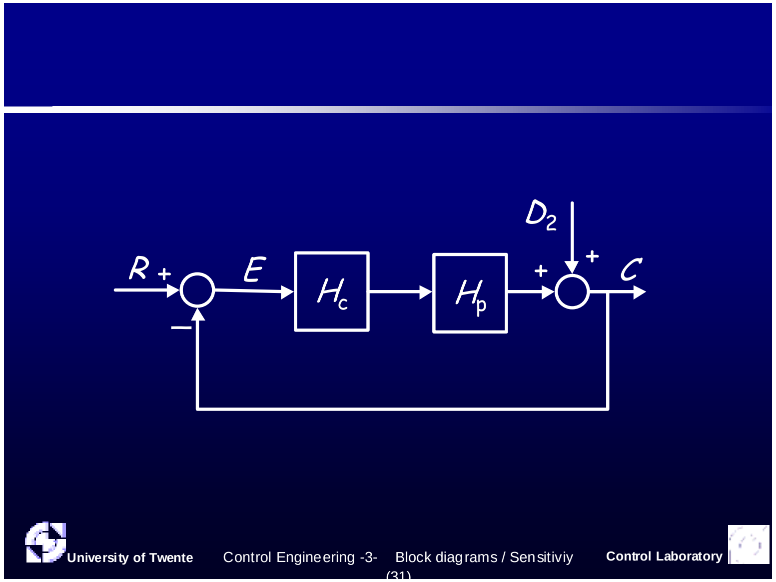
$$\text{if } H_c H_p \rightarrow \infty$$

$$\frac{C}{R} \approx \frac{H_c H_p}{H_c H_p H_m} = \frac{1}{H_m}$$



$H_m = 1$ is desired

Sensitivity



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$$\frac{C}{D_2} = \frac{1}{1+H_cH_p}$$

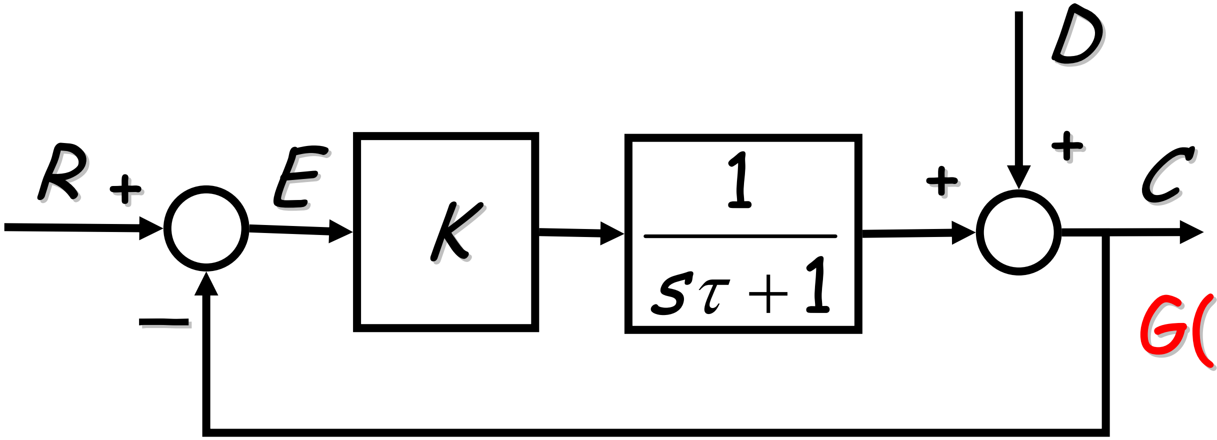
$$\frac{C}{R} = \frac{H_cH_p}{1+H_cH_p}$$

$$\boxed{\frac{1}{1+H_cH_p}} \boxed{+ \frac{H_cH_p}{1+H_cH_p}} = 1$$

Sensitivity

Complementary Sensitivity

Example (type 0)



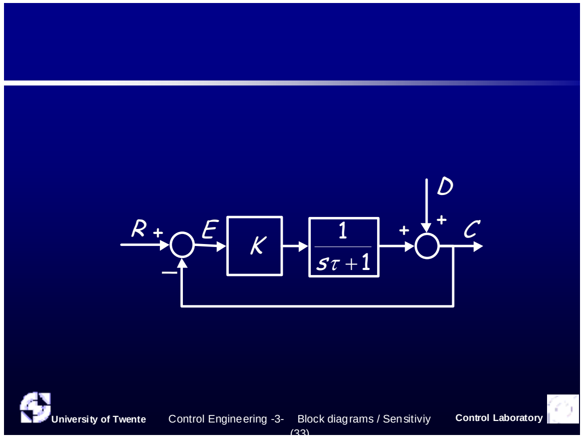
G(s) = first order

$$\frac{C}{D}=\frac{1}{1+\frac{K}{s\tau+1}}=\frac{1}{1+KG(s)}$$

pure integrators = 0
(type 0)

$$\frac{C}{D}=\frac{s\tau+1}{s\tau+1+K}=\left(\frac{1}{1+K}\right)\left(\frac{s\tau+1}{s\frac{\tau}{1+K}+1}\right)$$

Example (type 0)



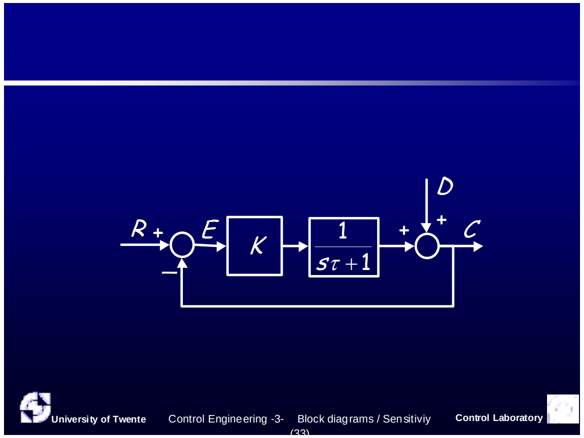
$$\frac{C}{D}=\left(\frac{1}{1+K}\right)\left(\frac{s\tau+1}{s\frac{\tau}{1+K}+1}\right)$$

if $D = 1/s$

$$\lim_{t\to\infty} c(t)=\lim_{s\to 0}\left[\frac{s}{s}\left(\frac{1}{1+K}\right)\left(\frac{s\tau+1}{s\frac{\tau}{1+K}+1}\right)\right]=\frac{1}{1+K}$$

ε_{ss}

Example (type 0)

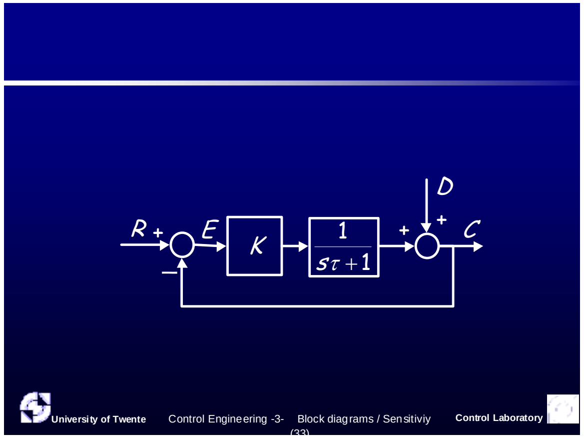


$$\frac{C}{D} = \left(\frac{1}{1+K}\right)\left(\frac{s\tau+1}{s\frac{\tau}{1+K}+1}\right)$$

if $D = 1/s$

$$\lim_{t\to0} c(t) = \lim_{s\to\infty}\left[\frac{s}{s}\left(\frac{1}{1+K}\right)\left(\frac{s\tau+1}{s\frac{\tau}{1+K}+1}\right)\right] = \frac{1}{1+K}\,\frac{\tau}{\frac{\tau}{1+K}} = 1$$

Example (type 0)



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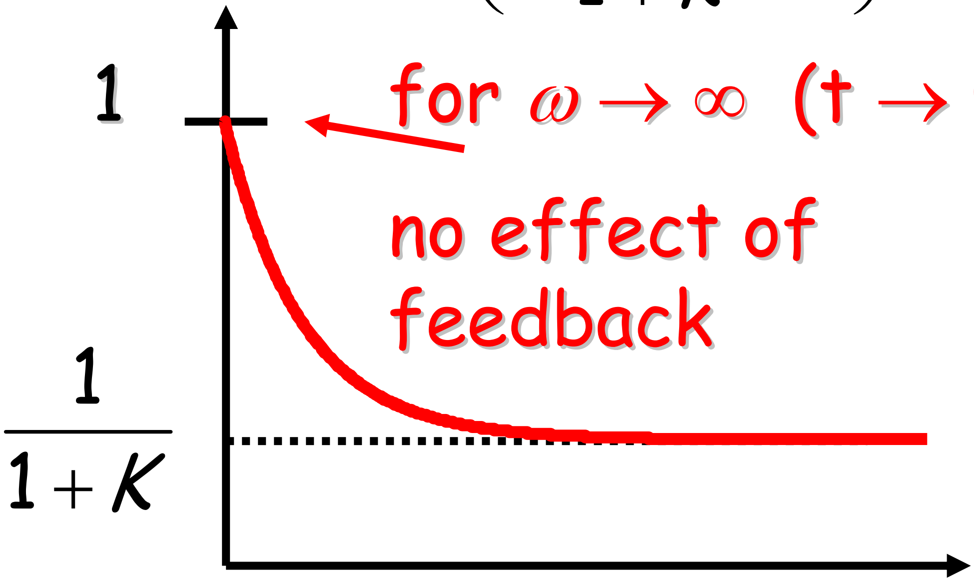
$$\frac{C}{D} = \left(\frac{1}{1+K}\right)\left(\frac{s\tau+1}{s\frac{\tau}{1+K}+1}\right)$$



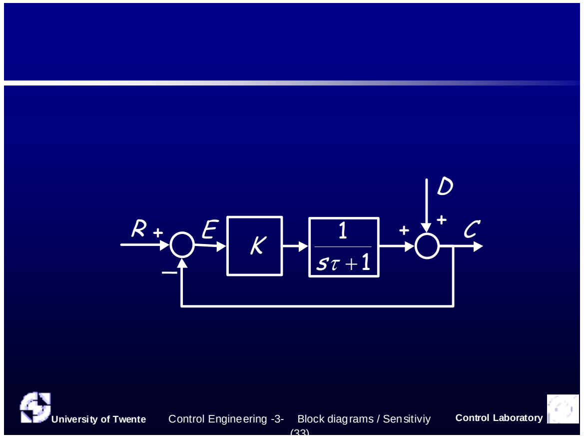
for $\omega \to \infty$ ($t \to 0$)

no effect of feedback

steady state error:

$$\varepsilon_{ss} = \frac{1}{1+K}$$

Demo 20-sim (type 0)



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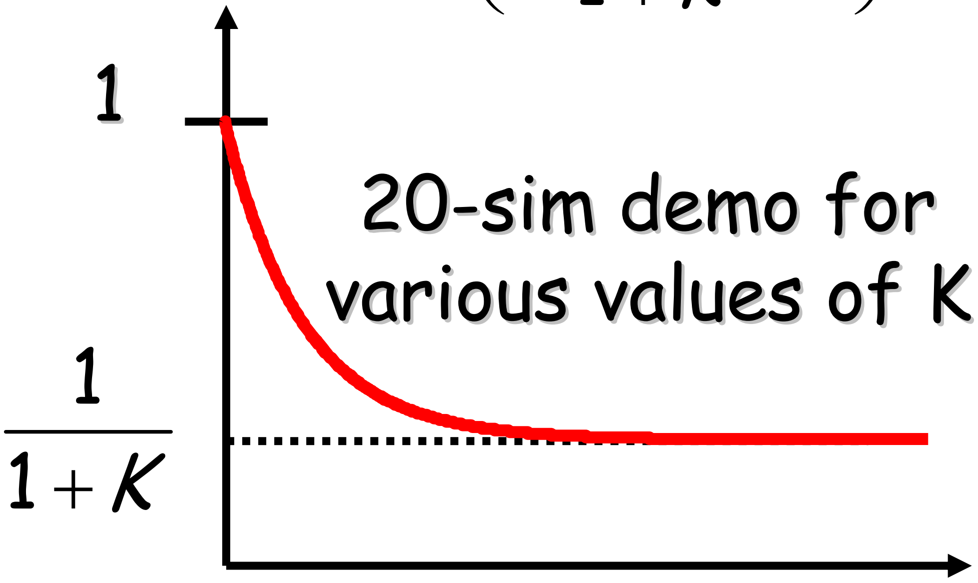
$$\frac{C}{D}=\left(\frac{1}{1+K}\right)\left(\frac{s\tau+1}{s\frac{\tau}{1+K}+1}\right)$$

steady state error:

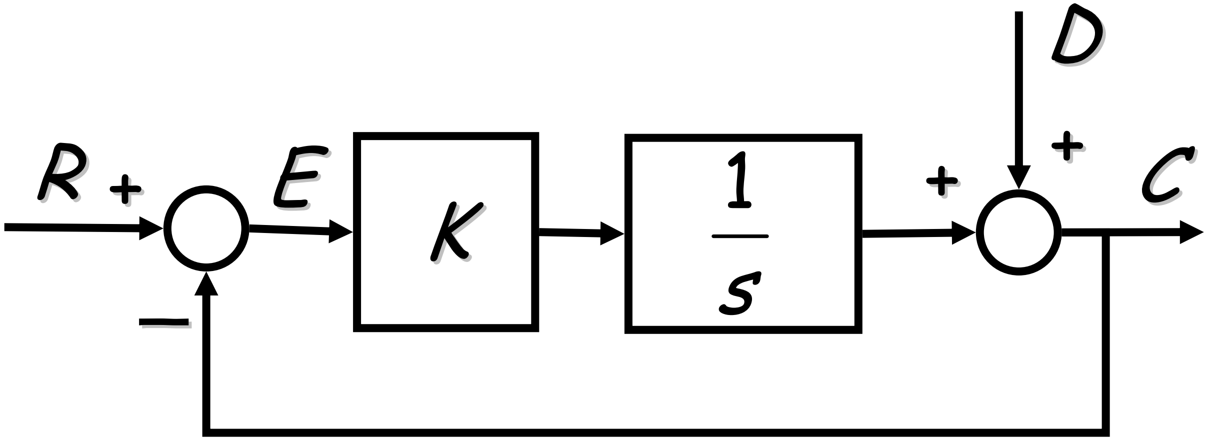
$$\varepsilon_{ss}=\frac{1}{1+K}$$

20-sim
demo



20-sim demo for various values of K

Example (type 1)



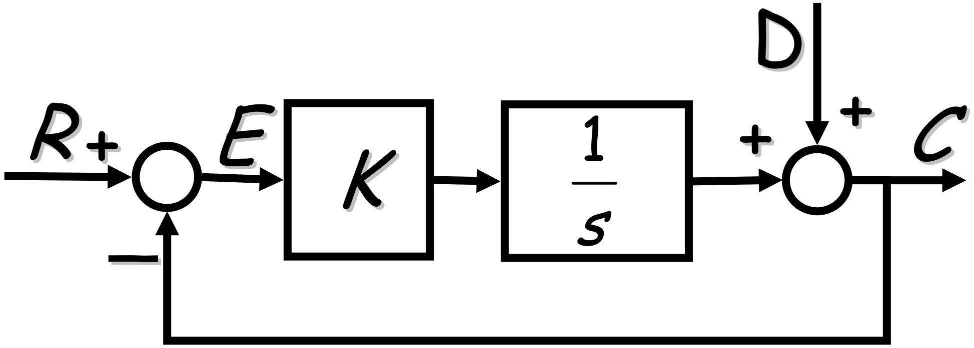
G(s) = first order

# of pure integrators = 1 (type 1)

$$\frac{C}{D} = \frac{1}{1+\dfrac{K}{s}} = \frac{1}{1+KG(s)}$$

$$\frac{C}{D} = \frac{s}{s+K} = \frac{1}{K}\left(\frac{s}{s\dfrac{1}{K}+1}\right)$$

Example (type 1)

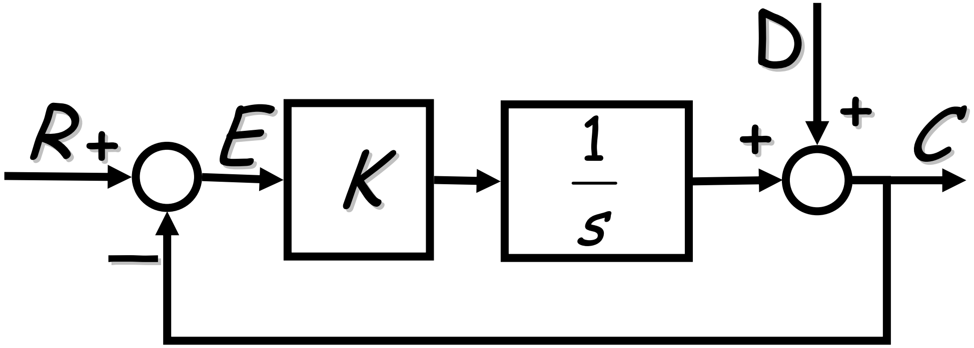


$$\frac{C}{D} = \frac{1}{K}\left(\frac{s}{s\frac{1}{K}+1} \right)$$

if $D = 1/s$

$$\lim_{t\to\infty} c(t) = \lim_{s\to 0}\left[\frac{s}{s}\frac{1}{K}\left(\frac{s}{s\frac{1}{K}+1} \right) \right] = 0$$

Example (type 1)

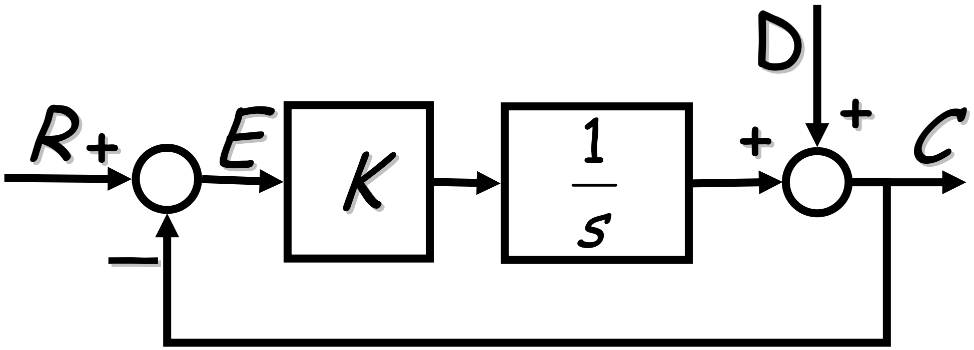


$$\frac{C}{D} = \frac{1}{K}\left(\frac{s}{s\frac{1}{K}+1}\right)$$

if $D = 1/s$

$$\lim_{t\to 0} c(t) = \lim_{s\to\infty}\left[\frac{s}{s}\frac{1}{K}\left(\frac{s}{s\frac{1}{K}+1}\right)\right] = \frac{1}{K}\frac{1}{\frac{1}{K}} = 1$$

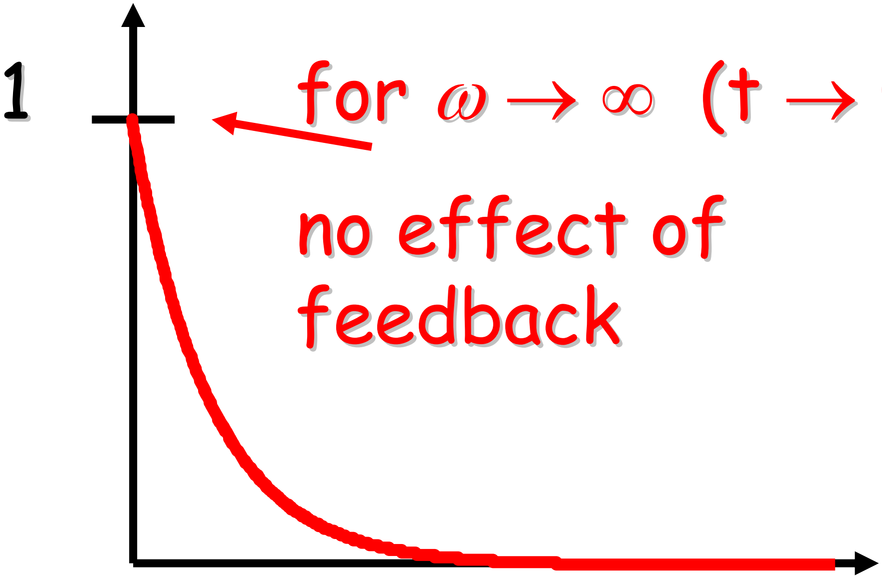
Example (type 1)



$$\frac{C}{D} = \frac{1}{K}\left(\frac{s}{s\frac{1}{K}+1}\right)$$

steady state error:

$$\varepsilon_{ss} = 0$$



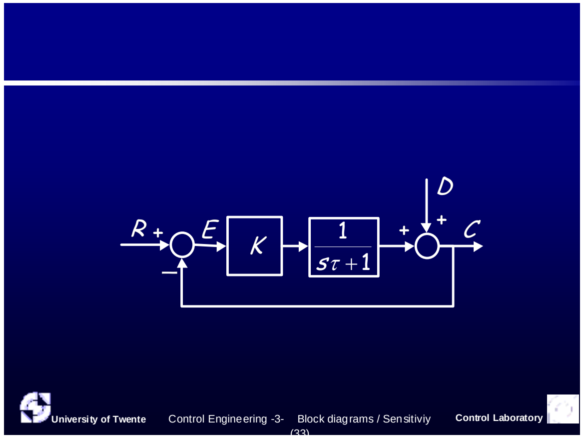
for $\omega \to \infty$ $(t \to 0)$

no effect of feedback

Steady State Errors

	type 0	type 1	type 2
step: $r(t) = C$	$\frac{1}{1+K}$	0	0
ramp: $r(t) = t$	?		

Ramp (type 0)



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$$\frac{C}{D} = \left(\frac{1}{1+K}\right)\left(\frac{s\tau+1}{s\frac{\tau}{1+K}+1}\right)$$

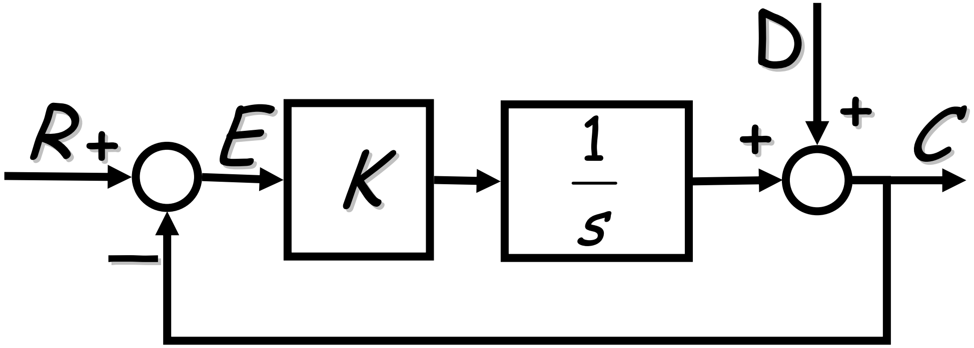
if $D = 1/s^2$

$$\lim_{t\to\infty} c(t) = \lim_{s\to 0}\left[\frac{s}{s^2}\left(\frac{1}{1+K}\right)\left(\frac{s\tau+1}{s\frac{\tau}{1+K}+1}\right)\right] = \lim_{s\to 0}\frac{1}{s}\to\infty$$

Steady State Errors

	type 0	type 1	type 2
step: $r(t) = C$	$\frac{1}{1+K}$	0	0
ramp: $r(t) = t$	∞	...	...

Ramp (type 1)



$$\frac{C}{D} = \frac{1}{K}\left(\frac{s}{s\frac{1}{K}+1}\right)$$

if $D = 1/s^2$

$$\lim_{t\to\infty} c(t) = \lim_{s\to 0}\left[\frac{s}{s^2}\frac{1}{K}\left(\frac{s}{s\frac{1}{K}+1}\right)\right] = \frac{1}{K}$$

Steady State Errors

	type 0	type 1	type 2
step: $r(t) = C$	$\frac{1}{1+K}$	0	0
ramp: $r(t) = t$	∞	$\frac{1}{K}$	0
$r(t) = t^2$	∞	∞	$\frac{1}{K}$

Excercises

- Determine the steady state errors of the last row of the previous slide

- Simulate type zero and type one systems with step inputs and ramp inputs